Zero Stability: A difference scheme for which perturbations remain bounded in the limit $h \to 0$ is said to be 0-stable.

- You can check 0-stability of an N step method

$$\sum_{k=0}^{N} a_k y_{j+k} = h \sum_{k=0}^{N} b_k f_{j+k} \qquad (*)$$

By determining the roots of the polynomial $P_N(\theta) = \sum_{k=0}^{N} a_k \theta^k$. If the roots of $P_n(\theta)$ are such that $|\theta| \leq 1$ and those for which $|\theta| = 1$ are simple then (*) is 0-stable.

Examples:

- The Euler Scheme is 0-stable since: $P_1(\theta) = \theta - 1 = 0$ has only one root on the unit disk.
- The Second order scheme $Y_{n+2} - 2Y_{n+1} + Y_n = \frac{h\lambda}{2}(Y_{n+2} - Y_n)$ is not 0-stable since $P_2(\theta) = (\theta - 1)^2 = 0$ has $\theta = 1$ as a double root.

Theorem: (Dahlquist) Consistency + 0-stability $\rightarrow$convergence. In particular, a 0-stable consistent method converges with the order of its truncation error.

Problem with practical use of the convergence theorem:

Eg: Consider the model problem $y' = \lambda y \qquad y(0) = 1 \rightarrow y(x) = e^{\lambda x}$.

Euler solution:

$$\begin{aligned} Y_n &= Y_{n-1} + h\lambda Y_{n-1}, \qquad Y_0 = 1 \\ &= (1 + h\lambda) Y_{n-1} \\ &\quad \uparrow \text{multiplication factor } G = 1 + \lambda h \\ &= (1 + h\lambda)^n Y_0 \qquad (*) \end{aligned}$$

Now let $n \to \infty$, $h \to 0$ in such a way that $nh = X$ a constant, then

$$\lim_{n \to \infty} (1 + h\lambda)^n = \lim_{n \to \infty} \exp\left\{ X\lambda \frac{\ln(1 + X\lambda/n)}{X\lambda/n} \right\} = e^{\lambda X}$$

so the method converges as $h \to 0$.

But in practice $h \neq 0$. Say $\lambda = -10$ and $h = 1$

$$Y_n = (-9)^n$$

which blows up and oscillates!

From (*) we observe that the solution will decay provided $|G(h\lambda)| = |1 + h\lambda| < 1$.

If λ is real then $-1 < 1 + h\lambda < 1 \Rightarrow \boxed{-2 < h\lambda < 0}$

Absolute Stability:

Recall: If $Re(\lambda) < 0$ then exact solutions of $y' = \lambda y$ decay with time i.e., $y_{\text{EX}}(x) = y_0 e^{-|\lambda| x}$.

Requirement of a difference scheme-asymptotic stability:

If $Re(\lambda) < 0$ then ideally we would like $|G(h\lambda)| < 1$ for all h. A method that satisfies this requirement is called asymptotically stable or A-stable.

Is FE A-stable? No.

Eg. $\lambda = -10$, $h = 1$, $|Y_n| \to \infty$. But the news is not all bad, if $Re(\lambda) < 0$ then there exist a range of values of h for which $|G(h\lambda)| < 1$.

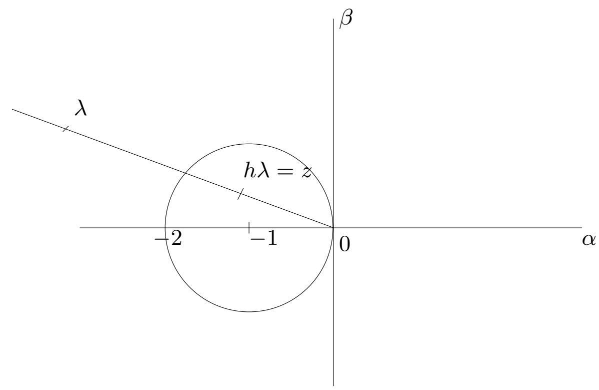
Stability Regions:

The set of points $z = h\lambda$ in the complex plane for which $|G(z)| < 1$.

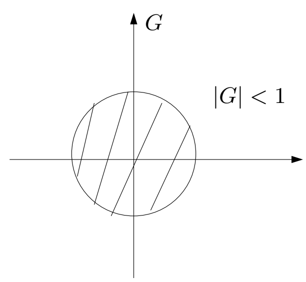
Stability region for FE:

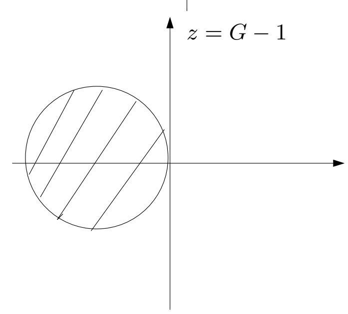
For what values of $z \in \mathbb{C}$ are $|G(z)| = |1 + z| < 1$.

Method 1.
$z = \alpha + i\beta \Rightarrow |1 + z|^2 = (1 + \alpha)^2 + \beta^2 < 1$



Method 2.
Using a conformal map:
$G = 1 + z \Rightarrow z = G - 1$.



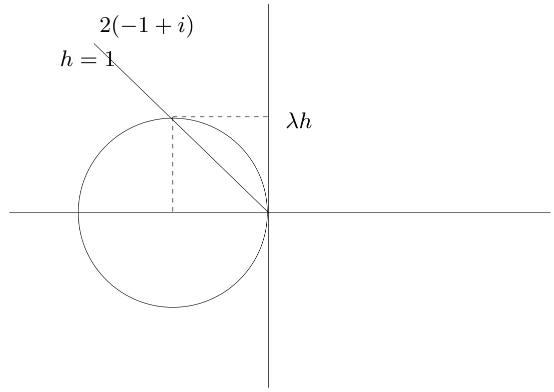


'**Usefully stable**': A method is usefully stable for a particular problem with eigenvalues λ_i and choice of timestep h if $h\lambda_i$ is in the stability region of the method for all λ_i.

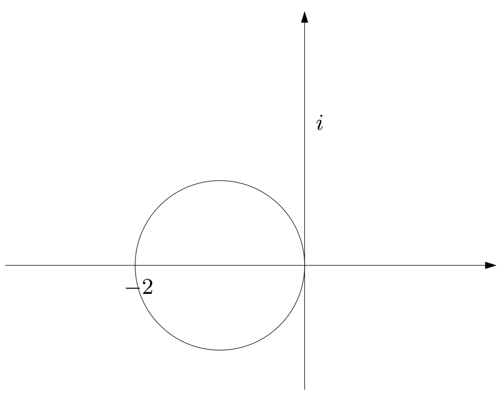
Eg: **Forward Euler** $\quad y' = \lambda y \quad y_0 = 1 \quad y = e^{\lambda x}$

1. $\lambda = -10$ for stability we require that $\lambda h : \; -2 < \lambda h < 0$

   $\therefore -2 < -h10 < 0 \therefore h < \frac{1}{5}$



2. $\lambda = 2(-1+i) = 2\sqrt{2}\, e^{i3\pi/4}$
   Point on BDY of disk is $h2(-1+i) = (-1+i)$
   $\therefore \quad h \le \frac{1}{2}$



3. $\lambda = i$ for no choice of h can $\lambda h = hi$ be brought into the stability region – so Euler's method will not be useful for oscillatory systems $y' = iy \Rightarrow y = e^{ix}$ which occur in models of wave phenomena.

Generalization of Euler's method

FE

‘1-Step’	Higher Order Difference	Multistep Methods	Predictor Corrector
• Improved Euler • Trapezoidal Scheme • Backward Euler • θ-Method • Modified Euler • RK	• Leapfrog	• Adams-Moulton • Adams-Bashforth • Backward Difference	• Improved Euler

Higher order DCE (Higher Order Difference, Multistep Methods)

Different approximations to $y'=f(x,y(x))$ (‘1-Step’, Higher Order Difference, Multistep Methods)

Different Philosophy (Predictor Corrector)

Schemes based on the Trapezoidal Rule:

Using the integral form of $y' = f(x, y(x))$

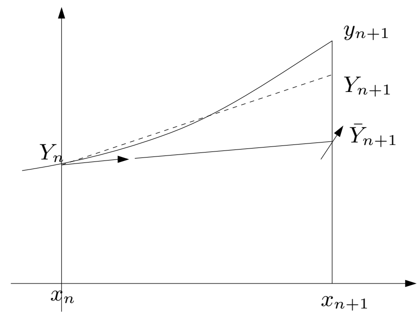
$$
\begin{aligned}
y(x_{n+1}) &= y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))dx \\
&= y(x_n) + \frac{h}{2}\left[f\left(x_n, y\left(x_n\right)\right) + f\left(x_{n+1}, y\left(x_{n+1}\right)\right)\right] + O(h^3) \qquad (*)
\end{aligned}
$$

There are a number of different ways we can choose to exploit $(*)$

(1)

The Improved Euler-Explicit/ Heun's method (RK-2)

$$
\left.\begin{array}{ll}
\text{Predictor} & \overline{Y}_{n+1} = Y_n + hf(x_n, Y_n) \\
\text{Corrector} & Y_{n+1} = Y_n + \frac{h}{2}\left[f\left(x_n, Y_n\right) + f\left(x_{n+1}, \overline{Y}_{n+1}\right)\right]
\end{array}\right\} \text{2 Stage}
$$



- Second order accurate
- Explicit
- If we keep replacing $\overline{Y}_{n+1}$ by Y_{n+1} until convergence, we obtain a 'predictor-corrector' method.

Truncation Error for the Improved Euler Scheme:

- **General Problem**: $y' = f(x, y)$, $y(0) = y_0$

$$
\begin{aligned}
Y_{k+1} &= Y_k + \frac{h}{2}\left[f(x_k, Y_k) + f(x_{k+1}, Y_k + hf(x_k, Y_k))\right] \\
T_k(h) &= \frac{y_{k+1} - y_k}{h} - \frac{1}{2}\left[f(x_k, y_k) + f(x_{k+1}, y_k + hf(x_k, y_k))\right] \\
&= \frac{y_k + hy'_k + \frac{h^2}{2}y''_k + \frac{h^3}{3!}y'''_k + \ldots - y_k}{h} - \frac{1}{2}\left[2f(x_k, y_k) + h(f_x + f_y f)|_{x_k} + O(h^2)\right] \\
&= \left(y'_k - f_k\right) + \frac{h}{2}\left(y''_k - (f_x + f_y f)\,|_{x_k}\right) + O(h^2) \\
\therefore\ & T_k(h) = O(h^2)
\end{aligned}
$$

- **On Model Problem – Simpler**: $y' = \lambda y \quad y(0) = y_0$

$$
\begin{aligned}
T_k(h) &= \frac{y_{k+1} - y_k}{h} - \frac{1}{2}\left[\lambda y_k + \lambda\left(y_k + h\lambda y_k\right)\right] \\
&= \frac{y_k + hy'_k + \frac{h^2}{2}y''_k + \frac{h^3}{3!}y'''_k + \ldots - y_k}{h} - \lambda y_k + h\frac{\lambda^2}{2}y_k \\
&= (y'_k - \lambda y_k) + \frac{h}{2}\left(y''_k - \lambda^2 y_k\right) + O(h^2)
\end{aligned}
$$

(2) **The Trapezoidal Scheme – 'Crank-Nicholson' Scheme – Implicit**

$$Y_{n+1} = Y_n + \frac{h}{2}\left[f(x_n, Y_n) + f(x_{n+1}, Y_{n+1})\right]$$

↑ Don't know this value – Implicit

We can solve this nonlinear equation (at each timestep) using Newton's method:

Let $g(Y_{n+1}) = Y_{n+1} - \frac{h}{2}\left[f(x_n, Y_n) + f(x_{n+1}, Y_{n+1})\right]$

$$Y_{n+1}^{k+1} = Y_{n+1}^{(k)} - g\left(Y_{n+1}^{(k)}\right) \Big/ g'\left(Y_{n+1}^{(k)}\right)$$

- Second Order Accurate $\left(f\left(x_{n+1}, y_{n+1}\right) = f\left(x_n + h, y_n + hy_n' + \frac{h^2}{2}y_n'' + \ldots\right)\right)$

$$\begin{aligned}
T_n(h) &= \frac{y_{n+1} - y_n}{h} - \frac{1}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}\right] \\
&= \frac{y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{3!}y_n''' + \ldots - y_n}{h} - \frac{1}{2}\left[f_n + f_n + h\left(f_x + f_y f\right)\Big|_{x_n} + O(h^2)\right] \\
&= \left(y_n' - f_n\right) + \frac{h}{2}\left(y_n'' - \left(f_x + f f_y\right)\Big|_{x_n}\right) + O(h^2).
\end{aligned}$$

- A quicker method to check the truncation error (useful for multistep methods but not very useful for multistage RK method).

$$\begin{aligned}
y_n' &= f_n \qquad y_{n+1}' = f_{n+1} \\
T_n(h) &= \frac{y_{n+1} - y_n}{h} - \frac{1}{2}\left\{f_n + f_{n+1}\right\} \\
&= \frac{y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + \ldots - y_n}{h} - \frac{1}{2}\left(y_n' + y_{n+1}'\right) \\
&= y_n' + \frac{h}{2}y_n'' + \frac{h^2}{6}y_n''' + \ldots - \frac{1}{2}\left(y_n' + y_n' + hy_n'' + \frac{h^2}{2}y_n'''\right) \\
&= \frac{h^2}{12}\, y_n''' + \ldots
\end{aligned}$$

↑ Error constant

Note: Improved Euler and the Trapezoidal Schemes are both second order accurate but the Improved Euler Scheme is explicit while the Trapezoidal Scheme is implicit. Then why would we bother using the Trapezoidal Scheme if we have to solve a nonlinear equation at each timestep?

Answer: Stability