Lecture 29: The heat equation with Robin BC

(Compiled 3 March 2014)

In this lecture we demonstrate the use of the Sturm-Liouville eigenfunctions in the solution of the heat equation. We first discuss the expansion of an arbitrary function $f(x)$ in terms of the eigenfunctions $\{\phi_n(x)\}$ associated with the Robin boundary conditions. This is a generalization of the Fourier Series approach and entails establishing the appropriate normalizing factors for these eigenfunctions. We then use the new generalized Fourier Series to determine a solution to the heat equation when subject to Robin boundary conditions.

Key Concepts: Eigenvalue Problems, Sturm-Liouville Boundary Value Problems; Robin Boundary conditions.

Reference Section: Boyce and Di Prima Section 11.1 and 11.2

29 Solving the heat equation with Robin BC

29.1 Expansion in Robin Eigenfunctions

In this subsection we consider a Robin problem in which $\ell = 1$, $h_1 \to \infty$, and $h_2 = 1$, which is a Case III problem as considered in lecture 30. In particular:

$$\phi'' + \mu^2 \phi = 0$$
$$\phi(0) = 0, \phi'(1) = -\phi(1)$$

$$\phi_n = \sin(\mu_n x), \quad \tan(\mu_n) = -\mu_n$$
$$\mu_n \sim \left(\frac{2n+1}{2}\right) \pi \quad \text{as } n \to \infty$$

Assume that we can expand $f(x)$ in terms of $\phi_n(x)$:

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$\int_0^1 f(x) \sin(\mu_n x) \, dx = c_n \int_0^1 [\phi_n(x)]^2 \, dx$$

$$= c_n \frac{1}{2} \left[1 + \cos^2 \mu_n\right]$$

Therefore

$$c_n = \frac{2}{\left[1 + \cos^2 \mu_n\right]} \int_0^1 f(x) \sin(\mu_n x) \, dx.$$
If \( f(x) = x \) then

\[
\int_0^1 x \sin(\mu_n x) \, dx = \frac{-\cos(\mu_n x)}{\mu_n} \bigg|_0^1 + \frac{1}{\mu_n} \int_0^1 \cos(\mu_n x) \, dx
\]

but \( -\mu_n \cos \mu_n = \sin \mu_n \)

Therefore

\[
e_n = \frac{4 \sin \mu_n}{\mu_n^2 [1 + \cos^2 \mu_n]} \quad (29.5)
\]

\[
f(x) = 4 \sum_{n=1}^{\infty} \frac{\sin \mu_n \sin(\mu_n x)}{\mu_n^2 [1 + \cos^2 \mu_n]} \quad (29.6)
\]

\[
f(x) = 4 \sum_{n=1}^{\infty} \frac{\sin \mu_n \sin(\mu_n x)}{\mu_n^2 [1 + \cos^2 \mu_n]} \quad (29.7)
\]

29.2 Solving the Heat Equation with Robin BC

Figure 1. Left: Initial and boundary conditions; Right: Solution profiles \( u(x, t) \) at various times

(b) Solution profiles \( u(x, t) \) at various times

\[
\begin{align*}
    u_t & = \alpha^2 u_{xx} & 0 < x < 1 \\
    u(0, t) & = 1 & u_x(1, t) + u(1, t) = 0 \\
    u(x, 0) & = f(x).
\end{align*}
\]

Look for a steady state solution \( v(x) \)

\[
\begin{cases}
    v''(x) = 0 \\
    v(0) = 1 & v'(1) + v(1) = 0
\end{cases}
\]

\[
v = Ax + B \quad v(0) = B = 1 \quad v'(x) = A \quad v'(1) + v(1) = A + (A + 1) = 0 \quad A = -1/2
\]

\[
A = -1/2
\]
Therefore
\[ v(x) = 1 - x/2. \] (29.13)

Now let \( u(x, t) = v(x) + w(x, t) \)
\[ u_t = w_t = \alpha^2(v'' + w_{xx}) \Rightarrow w_t = \alpha^2 w_{xx} \]
\[ 1 = u(0, t) = v(0) + w(0, t) = 1 + w(0, t) \Rightarrow w(0, t) = 0 \]
\[ 0 = u_x(1, t) + u(1, t) = \{ v'(1) + v(1) \} + w_x(1, t) + w(1, t) \Rightarrow w_x(1, t) + w(1, t) = 0 \]
\[ f(x) = u(x, 0) = v(x) + w(x, 0) \Rightarrow w(x, 0) = f(x) - v(x). \]

Let
\[ w(x, t) = X(x)T(t) \] (29.14)
\[ \frac{T'(t)}{\alpha^2 T(t)} = X'' = -\mu^2 \] (29.15)
\[ T(t) = e^{-\alpha^2 \mu^2 t} \] (29.16)
\[
\begin{align*}
X'' + \mu^2 X &= 0 \\
X(0) &= 0 \\
X'(1) + X(1) &= 0
\end{align*}
\]
\[ \{ \text{The } \mu_n \text{ are solutions of the transcendental equation: } \tan \mu_n = -\mu_n. \} \] (29.17)
\[ X_n(x) = \sin(\mu_n x) \] (29.18)
\[ w(x, t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \mu_n^2 t} \sin(\mu_n x) \] (29.19)

where
\[ f(x) - v(x) = w(x, 0) = \sum_{n=1}^{\infty} c_n \sin(\mu_n x) \] (29.20)
\[ \Rightarrow c_n = \frac{2}{1 + \cos^2 \frac{\pi}{\mu_n}} \int_0^1 [f(x) - v(x)] \sin(\mu_n x) \, dx \] (29.21)
\[ u(x, t) = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \mu_n^2 t} \sin(\mu_n x). \] (29.22)