Lecture 16: Bessel’s Inequality, Parseval’s Theorem, Energy convergence

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In this lecture we consider the counterpart of Pythagoras’ Theorem for functions whose square is integrable. Square integrable functions are associated with functions describing physical systems having finite energy. For a finite Fourier Series involving $N$ terms we derive the so-called Bessel Inequality, in which $N$ can be taken to infinity provided the function $f$ is square integrable. The Bessel Inequality is shown to reduce to an equality if and only if the Fourier Series $S_n(x)$ converges to $f$ in the energy norm. The result is known as Parseval’s Formula, which has profound consequences for the completeness of the Fourier Basis $\{1, \cos(\frac{n\pi x}{L}), \sin(\frac{n\pi x}{L})\}$. We see that Parseval’s Formula leads to a new class of sums for series of reciprocal powers of $n$.

Key Concepts: Convergence of Fourier Series, Bessel’s Inequality, Parseval’s Theorem, Plancherel theorem, Pythagoras’ Theorem, Energy of a function, Convergence in Energy, completeness of the Fourier Basis.

16 Bessel’s Inequality and Parseval’s Theorem:

16.1 Bessel’s Inequality

Definition 1 Let $f(x)$ be a function that is square-integrable on $[-L, L]$ i.e.,

$$\int_{-L}^{L} [f(x)]^2 \, dx < \infty,$$

in which case we write $f \in L^2[-L, L]$.

Consider the Fourier Series associated with $f(x)$, namely;

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) = S_\infty$$

Let

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right).$$

Now

$$[f(x) - S_N(x)]^2 = f^2(x) - 2f(x)S_n(x) + S_N^2(x)$$
Consider the least-square error defined to be

\[
E_2[f, S_N] = \frac{1}{L} \int_{-L}^{L} [f(x) - S_N(x)]^2 \, dx
\]

\[
= \frac{1}{L} \left\{ \int_{-L}^{L} f^2(x) \, dx - 2 \int_{-L}^{L} f(x) S_N(x) \, dx + \int_{-L}^{L} S_N^2(x) \, dx \right\}
\]

\[
= \frac{1}{L} \{ (f, f) - 2\langle f, S_N \rangle + \langle S_N, S_N \rangle \}
\]

Now

\[
\langle S_N, S_N \rangle = \frac{L}{-L} \left[ \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right]^2 \, dx
\]

\[
= \frac{a_0^2}{2} L + \sum_{n=1}^{N} a_n^2 \int_{-L}^{L} \cos^2 \left( \frac{n\pi x}{L} \right) \, dx + b_n^2 \int_{-L}^{L} \sin^2 \left( \frac{n\pi x}{L} \right) \, dx
\]

\[
= L \left[ \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2 \right]
\]

In addition,

\[
\langle f, S_N \rangle = \frac{L}{-L} \int_{-L}^{L} f(x) S_N(x) \, dx
\]

\[
= \frac{a_0}{2} \int_{-L}^{L} f(x) \, dx + \sum_{n=1}^{N} a_n \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx + b_n \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx
\]

\[
= \frac{a_0^2}{2} L + \sum_{n=1}^{N} a_n^2 L + b_n^2 L.
\]

Therefore

\[
E_2[f, S_N] = \frac{1}{L} \int_{-L}^{L} [f(x) - S_N(x)]^2 \, dx = \frac{1}{L} (f, f) - \left\{ \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2 \right\}
\]

Now since \( E_2[f, S_N] = \int_{-L}^{L} [f(x) - S_N(x)]^2 \, dx \geq 0 \) it follows that

\[
\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2 \leq \frac{1}{L} \int_{-L}^{L} f^2(x) \, dx = \frac{1}{L} (f, f) = E[f]
\]

where \( E[f] \) is known as the energy of the \( 2L \)-periodic function \( f \).
**Theorem 1** Bessel’s Inequality: Let \( f \in L^2[-L, L] \) then
\[
\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \leq \frac{1}{L} \int_{-L}^{L} f^2(x) \, dx
\]
in particular the series \( \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \) is convergent.

16.2 Bessel’s Inequality, Components of a Vector and Pythagoras’ Theorem

16.2.1 2D Analogue
Consider a 2D vector \( \tilde{f} \), which is decomposed into components in terms of two orthogonal unit vectors \( \hat{e}_1 \) and \( \hat{e}_2 \), i.e.
\[
\tilde{f} = a_1 \hat{e}_1 + a_2 \hat{e}_2
\]
Now
\[
|\tilde{f}|^2 = \tilde{f} \cdot \tilde{f} = (a_1 \hat{e}_1 + a_2 \hat{e}_2) \cdot (a_1 \hat{e}_1 + a_2 \hat{e}_2) = a_1^2 + a_2^2 \text{ since } \hat{e}_k \text{ are orthogonal unit vectors}
\]
Therefore \( |\tilde{f}|^2 = a_1^2 + a_2^2 \) which is Pythagoras’ Theorem.

16.2.2 3D Analogue
Suppose we wish to expand a 3-vector \( \tilde{f} \) in terms of a set of 2 basis vectors \( \{\hat{e}_1, \hat{e}_2\} \). Bessel’s Inequality assumes the form
\[
a_1^2 + a_2^2 \leq |\tilde{f}|^2
\]
Since the subspace span \( \{\hat{e}_1, \hat{e}_2\} \) (which represents a plane in \( \mathbb{R}^3 \)) does not include the whole of \( \mathbb{R}^3 \) the vector \( a_1 \hat{e}_1 + a_2 \hat{e}_2 \approx \tilde{f} \) represents the orthogonal projection of \( \tilde{f} \) onto span \( \{\hat{e}_1, \hat{e}_2\} \). If we include the third basis vector \( \hat{e}_3 \) in the basis, then the span \( \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \mathbb{R}^3 \). In this case the set \( \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \) are linearly independent and of full rank and thus span the complete space \( \mathbb{R}^3 \). \( \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \) are in this case said to form a complete set. In this case
\[
\tilde{f} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3
\]
and \( |\tilde{f}|^2 = a_1^2 + a_2^2 + a_3^2 \) so that Bessel’s Inequality assumes the form of an equality, which in this trivial case reduces to Pythagoras’ Theorem. For a set of functions, that are complete, the equivalent of Pythagoras’ Theorem is Parseval’s Theorem.
16.3 Parseval’s Theorem

**Theorem 2** (Parseval’s Identity) Let \( f \in L^2[-L, L] \) then the Fourier coefficients \( a_n \) and \( b_n \) satisfy Parseval’s Formula

\[
\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{L} \int_{-L}^{L} f^2(x) \, dx = E[f]
\]

If and only if

\[
\lim_{N \to \infty} \int_{-L}^{L} |f(x) - S_N(x)|^2 \, dx = 0.
\]

In this case the The Least Square Error assumes the form

\[
E_2[f, S_N] = \frac{1}{L} \int_{-L}^{L} |f(x) - S_N(x)|^2 \, dx = \frac{1}{L} \int_{-L}^{L} f^2(x) \, dx - \left( \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2 \right)
\]

\[
= \left( \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \right) - \left( \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2 \right)
\]

\[
= \sum_{n=N+1}^{\infty} a_n^2 + b_n^2
\]

(16.1)

16.3.1 Parseval’s Theorem for odd functions

**Theorem 3** (Parseval’s Identity for odd functions)

Let \( f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right) \) \( 0 < x < L \). Then

\[
\frac{2}{L} \int_{0}^{L} |f(x)|^2 \, dx = \sum_{n=1}^{\infty} b_n^2.
\]

Proof:

\[
\int_{0}^{L} |f(x)|^2 \, dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_m b_n \int_{0}^{L} \sin \left( \frac{m\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \, dx
\]

\[
= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_m b_n \cdot \delta_{mn} \cdot \frac{L}{2} = \frac{L}{2} \sum_{n=1}^{\infty} b_n^2.
\]

(16.2)

(16.3)

Example 16.1 Recall for \( x \in [0, 2] \), \( f(x) = x = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \left( \frac{n\pi x}{2} \right) \). Therefore

\[
\frac{2}{2} \int_{0}^{2} |f(x)|^2 \, dx = \left( \frac{4}{\pi} \right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

\[
\Rightarrow \frac{x^2}{3} \bigg|_0^2 = \left( \frac{4}{\pi} \right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

(16.4)
Fourier Series

Note: \[ \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \left( \frac{\pi^2}{6} \right) = \frac{\pi^2}{24}. \]

Also note that

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \]

\[ \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{24} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}. \]

Therefore

\[ \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{24} + \frac{\pi^2}{12} - \frac{\pi^2}{24} = \frac{\pi^2}{8}. \]  \quad (16.5)

For Fourier Sine Components:

\[ \frac{2}{L} \int_{0}^{L} (f(x))^2 \, dx = \sum_{n=1}^{\infty} b_n^2. \]  \quad (16.6)

**Example 16.2** Consider \( f(x) = x^2, -\pi < x < \pi. \)

The Fourier Series Expansion is:

\[ x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx). \]  \quad (16.7)

Let

\[
\begin{align*}
\cos \left( \frac{n \pi}{2} \right) &= 0, -1, 0, 1 \\
x = \frac{\pi}{2} \Rightarrow \frac{\pi^2}{4} &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \left( \frac{n \pi}{2} \right)
\end{align*}
\]

Therefore

\[ \frac{\pi^2}{12} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}. \]  \quad (16.8)

By Parseval's Formula:

\[ \frac{2}{\pi} \int_{0}^{x} dx = 2 \left( \frac{x^2}{\pi} \right)^2 + 16 \sum_{n=1}^{\infty} \frac{1}{n^4} \frac{9-5}{45} = \frac{4}{25} = \frac{8}{90}. \]  \quad (16.9)

Therefore

\[ \frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4} = \zeta(4), \]  \quad (16.10)

where \( \zeta \) is the Riemann Zeta Function defined by:

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + (i)\tau, \quad \sigma = \text{Re}[s] > 1 \]  \quad (16.11)