Lecture 12: Heat equation on a circular ring - full Fourier Series

(Compiled 3 March 2014)

In this lecture we use separation of variables to solve the heat equation subject on a thin circular ring with periodic boundary conditions. In this case we reduce the problem to expanding the initial condition function \( f(x) \) in an infinite series of both cosine and sine functions, which we refer to as the Full Range Fourier Series.

**Key Concepts:** Heat Equation; Periodic Boundary Conditions; separation of variables; Full Fourier Series.

### 12.1 Heat equation on a circular Ring - Full Fourier Series

**Physical Interpretation:** Consider a thin circular wire in which there is no radial temperature dependence, i.e., \( u(r, \theta) = u(\theta) \) so that \( \frac{\partial u}{\partial r} = 0 \). In this case the polar Laplacian reduces to

\[
\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \\
= \frac{\partial^2 u}{\partial (r\theta)^2}
\]  

(12.1)
and if we let $x = r \theta$ then $\frac{\partial^2 u}{\partial r^2} = u_{xx}$. In this case the heat distribution in the ring is determined by the following initial value problem with periodic boundary conditions

$$u_t = \alpha^2 u_{xx}$$  \hspace{1cm} (12.2)

BC: \begin{align*}
{u(-L,t)} &= {u(L,t)} \\
\frac{\partial u}{\partial x}(-L,t) &= \frac{\partial u}{\partial x}(L,t)
\end{align*} \hspace{1cm} \text{Periodic BC}

IC: \begin{align*}
u(x,0) &= f(x)
\end{align*}

Assume $u(x,t) = X(x)T(t)$.

As before: $\frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha^2 T(t)} = -\lambda^2$.

IVP: $\frac{T'(t)}{\alpha^2 T(t)} = -\lambda^2 \Rightarrow T(t) = e^{-\lambda^2 t}$.

$r = \frac{2L}{2\pi} = \frac{L}{\pi} = \text{Constant.}$

Eigenvalue Problem

\begin{align*}
X'' + \lambda^2 X &= 0 \\
X'(-L) &= X'(L)
\end{align*} \hspace{1cm} \text{look for } \lambda \text{ such that nontrivial } x \text{ can be found.}

\begin{align*}
X(x) &= A \cos(\lambda x) + B \sin(\lambda x) \\
X(-L) &= A \cos(\lambda L) - B \sin(\lambda L) = A \cos(\lambda L) + B \sin(\lambda L) = X(L) \\
\text{therefore } 2B \sin(\lambda L) &= 0 \\
X'(x) &= -A\lambda \sin(\lambda x) + B\lambda \cos(\lambda x) \\
X'(-L) &= -A\lambda \sin(\lambda L) + B\lambda \cos(\lambda L) = -A\lambda \sin(\lambda L) + B\lambda \cos(\lambda L) = X'(L) \\
\text{therefore } 2A\lambda \sin(\lambda L) &= 0
\end{align*} \hspace{1cm} (12.3)

Therefore $\lambda_n L = (n\pi) \hspace{0.5cm} n = 0, 1, \ldots.$

Solutions to (12.2) that satisfy the BC are thus of the form

$$u_n(x,t) = e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} \left\{A_n \cos \left(\frac{n\pi x}{L}\right) + B_n \sin \left(\frac{n\pi x}{L}\right)\right\}. \hspace{1cm} (12.4)$$

Superposition of all these solutions yields the general solution

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \left\{A_n \cos \left(\frac{n\pi x}{L}\right) + B_n \sin \left(\frac{n\pi x}{L}\right)\right\} e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t}. \hspace{1cm} (12.5)$$

In order to match the IC we have

$$f(x) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \left(\frac{n\pi x}{L}\right) + B_n \sin \left(\frac{n\pi x}{L}\right). \hspace{1cm} (12.6)$$
Fourier Series

As before we obtain expressions for the $A_n$ and $B_n$ by projecting $f(x)$ onto $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{n\pi x}{L}\right)$.

$$\int_{-L}^{L} f(x) \left\{ \sin\left(\frac{m\pi x}{L}\right) \right\} dx = A_0 \int_{-L}^{L} \left\{ \sin\left(\frac{m\pi x}{L}\right) \right\} dx$$

$$+ \sum_{n=1}^{\infty} A_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \left\{ \sin\left(\frac{m\pi x}{L}\right) \right\} dx$$

$$+ \sum_{n=1}^{\infty} B_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \left\{ \cos\left(\frac{m\pi x}{L}\right) \right\} dx. \quad (12.7)$$

As in the previous example we use the orthogonality relations:

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = L\delta_{mn}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = L\delta_{mn} \quad m \neq n$$

$$= 2L \quad m = n = 0$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0 \quad \forall m, n. \quad (12.8)$$

Plugging these orthogonality conditions into (12.6) we obtain

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx = \text{average value of } f(x) \text{ on } [-L, L]$$

$$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \quad (12.9)$$

Observations:

1. (12.6) and (12.9) represent the full Fourier Series Expansion for $f(x)$ on the interval $[-L, L]$.

2. By defining $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 2A_0 & \text{if } n = 0, \\ A_n & \text{else} \end{cases}$ and $b_n = B_n$ the Fourier Series (12.6) is often written in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right). \quad (12.10)$$