Lecture 11: Fourier Cosine Series

(Compiled 4 August 2017)

In this lecture we use separation of variables to solve the heat equation subject to Neumann boundary conditions. In this case we reduce the problem to expanding the initial condition function $f(x)$ in an infinite series of cosine functions - known as the Fourier Cosine Series.

**Key Concepts:** Heat Equation; Neumann Boundary Conditions; separation of variables; Fourier Cosine Series.

11 The heat equation subject to Homogenous Neumann Boundary Conditions

We consider the heat equation subject to the following initial and boundary conditions:

\begin{align*}
\frac{\partial u(0, t)}{\partial x} &= 0 \\
u_t &= \alpha^2 u_{xx} \\
\frac{\partial u(L, t)}{\partial x} &= 0 \\
u(x, 0) &= f(x)
\end{align*}

\textbf{Figure 1.} Consider a conducting bar with thermal conductivity $\alpha^2$ that has an initial temperature distribution $u(x, 0) = f(x)$ and whose endpoints are insulated.

Heat Equation : $u_t = \alpha^2 u_{xx}, \ 0 < x < L$ \quad (11.1)
Boundary Conditions : $\frac{\partial u(0, t)}{\partial x} = 0 = \frac{\partial u(L, t)}{\partial x}$ \quad (11.2)
Initial Condition : $u(x, 0) = f(x)$ \quad (11.3)
11.1 Separation of Variables - Fourier sine Series:

Consider the heat conduction in an insulated rod whose endpoints are insulate for all time and within which the initial temperature is given by \( f(x) \) as shown in figure 1.

**Fourier’s Guess:**

\[
\begin{align*}
  u(x, t) &= X(x)T(t) \\
  u_t &= X(x)T'(t) = \alpha^2 u_{xx} = \alpha^2 X''(x)T(t)
\end{align*}
\]

\(\Rightarrow\alpha^2 XT:\)

\[
\frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha^2 T(t)} = \text{Constant} = -\lambda^2.
\]

**Time equation**

\[
\begin{align*}
  \dot{T}(t) &= -\alpha^2 \lambda^2 T(t) \quad \frac{dT}{T} = -\alpha^2 \lambda^2 \, dt \\
  \ln|T| &= -\alpha^2 \lambda^2 t + c \\
  T(t) &= D e^{-\alpha^2 \lambda^2 t}.
\end{align*}
\]

**Case I: Spatial equation assuming that \( \lambda \neq 0 \):**

\[
\begin{align*}
  X''(x) + \lambda^2 X(x) &= 0 \\
  \text{Guess} \quad X(x) &= e^{rx} \Rightarrow (r^2 + \lambda^2)e^{rx} = 0 \\
  r &= \pm \lambda i
\end{align*}
\]

\[
X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}
\]

\[
X' = A \cos \lambda x + B \sin \lambda x
\]

Now impose the boundary conditions:

\[
\begin{align*}
  0 &= X'(0) = B \
  0 &= X'(L) = -A \lambda \cos \lambda L + B \lambda \sin \lambda L
\end{align*}
\]

Now substitute the solution from (11.8) and use the fact that we have assumed that \( \lambda \neq 0 \)

\[
\begin{align*}
  0 &= X'(0) = -A \lambda \cos \lambda L + B \lambda \sin \lambda L \
  0 &= X'(L) = A \lambda \sin \lambda L + B \lambda \cos \lambda L
\end{align*}
\]

Therefore for the case \( \lambda \neq 0 \) we have the countably infinite set of eigenvalues and eigenfunctions

\[
\lambda_n = \left( \frac{n \pi}{L} \right) \quad n = 1, 2, \ldots
\]

**Case II: Spatial equation assuming that \( \lambda = 0 \):**

In this case the spatial ODE reduces to

\[
X''(x) = 0
\]

which has a general solution

\[
\begin{align*}
  X(x) &= A_1 + B x \\
  X'(x) &= B
\end{align*}
\]

Now imposing the boundary conditions

\[
\begin{align*}
  0 &= X'(0) = B \Rightarrow B = 0 \\
  0 &= X'(L) = B \Rightarrow B = 0
\end{align*}
\]
Fourier Series

The complete set of eigenvalues and eigenfunctions are thus:

$$\lambda_n = \left(\frac{n\pi}{L}\right) \quad n = 0, 1, 2, \ldots \text{ and } \quad X_0(x) = 1, \quad X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots. \quad (11.14)$$

Thus $$u_n(x, t) = e^{-\alpha^2\left(\frac{n\pi x}{L}\right)^2} \cos\left(\frac{n\pi x}{L}\right) \quad n = 0, 1, 2, \ldots$$

are all solutions of $$u_t = \alpha^2 u_{xx}$$ that satisfy the boundary conditions (11.2). (11.15)

Since (11.1) is linear, a linear combination of solutions is again a solution. Thus the most general solution is for the form

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha^2\left(\frac{n\pi x}{L}\right)^2}. \quad (11.16)$$

What about the initial condition $$u(x, 0) = f(x)$$? If we let $$t = 0$$ in (11.16), then to complete the solution process we are reduced to determining the coefficients $$A_n$$ in the series

$$u(x, 0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right). \quad (11.17)$$

As in the last lecture we use the inner product $$\langle \cdot, \cdot \rangle$$ to project $$f(x)$$ onto the basis functions in the series:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \quad (11.18)$$

$$\langle f, \cos\left(\frac{k\pi x}{L}\right) \rangle = \int_0^L f(x) \cos\left(\frac{k\pi x}{L}\right) dx = A_0 \int_0^L \cos\left(\frac{k\pi x}{L}\right) dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{k\pi x}{L}\right) dx \quad (11.19)$$

Recall the identity $$\cos(A) \cos B = \frac{1}{2} \{\cos(A - B) + \cos(A + B)\}$$. Therefore

$$J_{nk} = \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{k\pi x}{L}\right) dx$$

$$= \frac{1}{2} \int_0^L \cos(n-k)\frac{\pi x}{L} + \cos(n+k)\frac{\pi x}{L} dx \quad n \neq k$$

$$= \frac{1}{2} \left[ \frac{\sin(n-k)\pi x/L}{(n-k)\pi/L} + \frac{\sin(n+k)\pi x/L}{(n+k)\pi/L} \right]_0^L$$

$$= 0 \quad (11.20)$$

$$J_{nn} = \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} \int_0^L 1 + \cos\left(\frac{2n\pi x}{L}\right) dx$$

$$= \frac{L}{2} \left[ x + \frac{\sin(2n\pi x/L)}{2n\pi/L} \right]_0^L$$

$$J_{00} = \int_0^L 1 dx = L$$
Substituting these integrals into (11.19) we obtain the following expressions for the Fourier Coefficients $A_k$

\[ A_0 = \frac{1}{L} \int_0^L f(x) \, dx. \quad (11.21) \]

\[ A_k = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{k\pi x}{L} \right) \, dx. \quad (11.22) \]

Finally the solution of the initial boundary value problem (11.1) is

\[ u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{L} \right) e^{-\alpha^2 \left( \frac{n\pi}{L} \right)^2 t}. \quad (11.23) \]

where $A_n$ are defined in (11.21)-(11.22). We observe that as $t \to \infty$ it follows that $u(x, t) \to A_0$, which is just the average value of the initial heat $f(x)$ distributed in the bar as can be seen from (11.21). This is consistent with physical intuition.

It is sometimes convenient to re-define the Fourier coefficients as follows:

\[ a_0 = 2A_0 \]

\[ a_k = A_k, \quad k = 1, 2, \ldots \]

so that the $a_k$ assume the unified form $a_k = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{k\pi x}{L} \right) \, dx \quad k = 0, 1, 2, \ldots \quad (11.24) \]

In terms of the new coefficients $a_k$ defined in (11.24) the Fourier expansion for the initial condition function $f(x)$ is of the form

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right). \quad (11.25) \]

while the solution of the heat equation (11.1) is of the form

\[ u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right) e^{-\alpha^2 \left( \frac{n\pi}{L} \right)^2 t}. \quad (11.26) \]

**Example 11.1** Fourier Cosine Expansion: Determine the Fourier coefficients $a_k$ for the function

\[ f(x) = x, \quad 0 < x < 1 = L \quad (11.27) \]

and use the resulting Fourier Cosine expansion to prove the identity

\[ \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \ldots + \frac{1}{(2k+1)^2} + \ldots \]

Solution:

\[ a_0 = 2 \int_0^1 x \, dx = 2 \left[ \frac{x^2}{2} \right]_0^1 = 1 \]

\[ a_n = 2 \int_0^1 x \cos(n\pi x) \, dx = 2 \frac{(-1)^n - 1}{n^2 \pi^2} = \begin{cases} -\frac{1}{n^2 \pi^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \]
substituting these expressions for the $a_n$ into (11.25), we obtain

$$f(x) = x = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos((2k+1)\pi x)$$

(11.28)

To obtain the required identity we set $x = 1$ in and rearrange terms. The partial sums are shown in figure 2.

Figure 2. These figures show the partial sums of the Fourier Cosine Series

In figure 3 we plot the same graphs but on a larger domain than $[0, L] = [0, 1]$.

Figure 3. These figures show the partial sums of the Fourier Cosine Series
MATLAB Code:

```matlab
% fourier cos example
clear;clf;dx=0.001;dt=0.001;
x=-2:dx:2;xr=0:dx:1;nterms=10;ntime=100;
for nt=1:ntime
    t = (nt-1)*dt;
    for n=1:nterms
        K=0:1:n;
        u(:,n+1)=0.5+4*(cos(pi*(2*K+1)'*x)'*(exp(-pi^2*(2*K+1).^2*t)./(2*K+1).^2)')/pi^2;
        plot(x',u(:,n+1),'r-',xr',xr','k-','linewid th',2);
        axis=ax;axis=[0 1 0 1.2];axis(ax);
        tit=[num2str(n+1),' terms of the Fourier Series '];title(tit);xlabel('x');ylabel('u(x,t), f(x)=x');pause(.01)
    end
    if mod(nt,5)==0,pause(.02);end
end
```