Lecture 1: Review of methods to solve Ordinary Differential Equations

(Compiled 26 January 2018)

In this lecture we will briefly review some of the techniques for solving First Order ODE and Second Order Linear ODE, including Cauchy-Euler/Equidimensional Equations

**Key Concepts:** First order ODEs: Separable and Linear equations; Second Order Linear ODEs: Constant Coefficient Linear ODE, Cauchy-Euler/Equidimensional Equations.

### 1 First Order ordinary Differential Equations:

#### 1.1 Separable Equations:

\[
\frac{dy}{dx} = P(x)Q(y) \tag{1.1}
\]

\[
\int \frac{dy}{Q(y)} = \int P(x) \, dx + C
\]

Example 1:

\[
\frac{dy}{dx} = \frac{4y}{x(y - 3)}
\]

\[
\left(\frac{y - 3}{y}\right) \, dy = \frac{4}{x} \, dx
\]

\[
y - 3 \ln |y| = 4 \ln |x| + C
\]

\[
y = \ln(x^4y^3) + C
\]

\[
Ax^4y^3 = e^y
\]

#### 1.2 Linear First Order equations - The Integrating Factor:

\[
y'(x) + P(x)y = Q(x) \tag{1.3}
\]

Can we find a function \( F(x) \) to multiply (4.3) by in order to turn the left hand side into a derivative of a product:

\[
Fy' + FPy = FQ \tag{1.4}
\]

\[
(Fy)' = Fy' + F'y = FQ \tag{1.5}
\]
2

So let \( F' = FP \) which is a separable Eq.

\[
\frac{dF}{F(x)} = P(x) \, dx \Rightarrow \int \frac{dF}{F} = \int P(x) \, dx + C
\]

Therefore \( \ln F = \int P(x) \, dx + C \) \hspace{1cm} (1.6)

or \( F = Ae^{\int P(x) \, dx} \) \hspace{1cm} choose \( A = 1 \)

\( F = e^{\int P(x) \, dx} \) \hspace{1cm} integrating factor

Therefore

\[
e^{\int P(x) \, dx} y' + e^{\int P(x) \, dx} P(x) y = e^{\int P(x) \, dx} Q(x)
\]

\[
(\text{e}^{\int P(x) \, dx} y)' = e^{\int P(x) \, dx} Q(x)
\]

\[
y(x) = e^{-\int P(x) \, dx} \left\{ \int e^{\int P(t) \, dt} Q(x) \, dx + C \right\}
\] \hspace{1cm} (1.7)

Example 2:

\[
y' + 2y = 0
\]

\[
F(x) = e^{2x} \Rightarrow e^{2x} y' + e^{2x} 2y = (e^{2x} y)' = 0
\]

\[
e^{2x} y = c
\]

\[
y(x) = Ce^{-2x}
\]

Example 3: Solve

\[
\frac{dy}{dx} + \cot(x) y = 5e^{\cos x}, \quad y(\pi/2) = -4
\]

\[
P(x) = \cot x \quad Q(x) = 5e^{\cos x}
\]

\[
F(x) = e^{\int \cot x \, dx} = e^{\ln(\sin x)} = \sin x
\] \hspace{1cm} (1.10)

Therefore \( \sin(x) y' + \cos(x) y = (\sin(x) y)' = 5e^{\cos x} \sin x \)

\[
sin(x) y = -5e^{\cos x} + C
\]

\[
y(x) = \frac{-5e^{\cos x} + C}{\sin x}
\] \hspace{1cm} (1.11)

\[-4 = y(\pi/2) = -\frac{5C}{4} \Rightarrow C = 1
\]

Therefore \( y(x) = \frac{1-5e^{\cos x}}{\sin x} \)

2 Second Order Constant Coefficient Linear Equations:

\[
Ly = ay'' + by' + cy = 0
\]

Guess \( y = e^{rx} \quad y' = re^{rx} \quad y'' = r^2e^{rx} \)

\[
Ly = [ar^2 + br + c]e^{rx} = 0 \text{ provided } r = 0
\]

Indicial Eq.:

\[
g(r) = ar^2 + br + c = 0 \quad r_{1,2} = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}
\]

or \( g(r) = a(r - r_1)(r - r_2) = 0 \) \hspace{1cm} (2.1)

Case I: \( \Delta = b^2 - 4ac > 0, \quad r_1 \neq r_2, \quad y(x) = c_1 e^{r_1x} + c_2 e^{r_2x} \) is the general solution.
Case II: $\Delta = 0$, $r_1 = r_2$, repeated roots $Ly = a(r - r_1)^2e^{rx} = 0$. In this case obtain only one solution $y(x) = e^{r_1x}$.

How do we get a second solution?

**First Method: Perturbation of the double root:** Consider a small perturbation (see figure 1 a) to the double root case, such that $g(r) = a(r - (r_1 - \epsilon))(r - (r_1 + \epsilon)) = a[(r - r_1)^2 - \epsilon^2] \approx a(r - r_1)^2$. In this case the two, very close but distinct, roots of $g(r) = 0$ are given by:

$$r = r_1 + \epsilon \text{ and } r = r_1 - \epsilon$$ (2.2)

Now since we still have two distinct roots in this perturbed case, the general solution is:

$$y(x) = c_1 e^{(r_1+\epsilon)x} + c_2 e^{(r_1-\epsilon)x}$$ (2.3)

Now choosing a special solution by selecting $c_1 = \frac{1}{2\epsilon} = -c_2$, and we obtain a family of solutions that depend on the small parameter $\epsilon$ (see figure 1 b):

$$y(x, \epsilon) = \frac{e^{(r_1+\epsilon)x} - e^{(r_1-\epsilon)x}}{2\epsilon} \approx \left| \frac{\partial}{\partial r} e^{rx} \right|_{r=r_1}$$ (2.4)

Now taking the limit as $\epsilon \to 0$ by making use of L'Hospital’s Rule, we obtain the following limiting solution:

$$y(x, \epsilon) = e^{r_1x} \left( \frac{e^{\epsilon x} - e^{-\epsilon x}}{2\epsilon} \right)_{\epsilon \to 0} x e^{r_1x} = \left| \frac{\partial}{\partial r} e^{rx} \right|_{r=r_1}$$ (2.5)

**Second Method: taking the derivative with respect to $r$:** From (2.4) and (2.5) we see that the new solution $xe^{r_1x}$ was obtained by taking the derivative of $y(x, r) = e^{rx}$ with respect to $r$ and then making the substitution $r = r_1$. This
is, in fact, a general procedure that we will use later in the course. To see why this procedure works, let

\[ y(r, x) = e^{rx} \]

\[ Ly(r, x) = (r - r_1)^2e^{rx} \]

\[ L \left[ \frac{\partial y}{\partial r}(r, x) \right]_{r=r_1} = \left[ 2a(r - r_1)e^{rx} + a(r - r_1)^2xe^{rx} \right]_{r=r_1} = 0 \tag{2.6} \]

Thus, to summarize, the general solution for the case of a double root is:

\[ y(x) = c_1e^{r_1x} + c_2xe^{r_1x} \tag{2.7} \]

Case III: Complex Conjugate Roots: \( \Delta = b^2 - 4ac < 0 \)

\[ r = \frac{-b \pm \sqrt{4ac - b^2}}{2a} = \lambda \pm i\mu \]

\[ y(x) = c_1e^{\lambda x}e^{i\mu x} + c_2e^{\lambda x}e^{-i\mu x} \tag{2.8} \]

\[ = e^{\lambda x} [A \cos \mu x + B \sin \mu x] . \]

Example 4:

\[ Ly = y'' + y' - 6y = 0 \]

\[ y = e^{rx}(r^2 + r - 6) = (r + 3)(r - 2) = 0 \]

\[ y(x) = c_1e^{-3x} + c_2e^{2x} \tag{2.9} \]

Example 5:

\[ Ly = y'' + 6y' + 9y = 0 \]

\[ y = e^{rx}(r + 3)^2 = 0 \]

\[ y(x) = c_1e^{-3x} + c_2e^{-3x} \tag{2.10} \]

Example 6:

\[ Ly = y'' - 4y' + 13y = 0 \]

\[ y = e^{rx} : r^2 - 4r + 13 = 0 \]

\[ r = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i \]

\[ Therefore \ y(x) = e^{2x} [A \cos 3x + B \sin 3x] . \tag{2.11} \]

3 Cauchy/Euler/Equidimensional Equations:

\[ Ly = x^2y'' + \alpha xy' + \beta y = 0. \tag{3.1} \]

Aside: Note if we let \( t = \ln x \) or \( x = e^t \) then

\[ \frac{d}{dx} = \frac{d}{dt} \frac{d}{dx} = \frac{d}{dt} \]

\[ \frac{d^2}{dt^2} = x \frac{d}{dx} \left( x \frac{d}{dx} \right) = x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} \frac{d^2}{dx^2} = \frac{d^2}{dt^2} - \frac{d}{dt} \]

\[ Therefore \quad \ddot{y} - \dot{y} + \alpha \dot{y} + \beta y = 0 \]

\[ \ddot{y} + (\alpha - 1) \dot{y} + \beta y = 0 \tag{3.3} \]

\[ y = e^{rt} \Rightarrow r^2 + (\alpha - 1)r + \beta = 0 \quad \text{Characteristic Eq.} \]
Review of methods to solve Ordinary Differential Equations

Back to (3.1): Guess \( y = x^r, \ y' = rx^{r-1}, \) and \( y'' = r(r-1)x^{r-2}. \)

Therefore \( \{ r(r-1) + \alpha r + \beta \} x^r = 0 \)
\[ f(r) = r^2 + (\alpha - 1)r + \beta = 0 \quad \text{as above.} \] (3.4)

\[
r_{\pm} = \frac{1 - \alpha \pm \sqrt{(\alpha - 1)^2 - 4\beta}}{2}
\] (3.5)

Case 1: \( \Delta = (\alpha - 1)^2 - 4\beta > 0 \) Two Distinct Real Roots \( r_1, r_2. \)

\[ y = c_1x^{r_1} + c_2x^{r_2} \] (3.6)

If \( r_1 \) or \( r_2 < 0 \) then \( |y| \to \infty \) as \( x \to 0. \)

Case 2: \( \Delta = 0 \) Double Root \( (r - r_1)^2 = 0. \)

We obtain only one solution in this case:

\[ y = c_1x^{r_1} \] (3.7)

To get a second solution we use second method introduced above, in which we differentiate with respect to the parameter \( r: \)

\[
\frac{\partial}{\partial r} L[x^r] = L \left[ \frac{\partial}{\partial r} x^r \right] = L[ x^r \log x ]
\]
\[
\frac{\partial}{\partial r} \{ f(r)x^r \} = f'(r)x^r + f(r)x^r \log x = 0 \quad \text{since } f(r) = (r - r_1)^2.
\] (3.8)

General Solution: \( y(x) = (c_1 + c_2 \log x) x^{r_1}. \)

Check:

\[
L(x^{r_1} \log x) = x^2(x^r \log x)^{''} + \alpha x(x^r \log x)^{'} + \beta (x^r \log x) -
\]
\[
= x^2 \left[ r(r-1)x^r \log x + rx^{r-2} + (r-1)x^{r-2} \right] + \alpha x \left[ rx^{r-1} \log x + x^{r-1} \right] + \beta (x^r \log x)
\]
\[
= \{ r^2 + (\alpha - 1)r + \beta \} x^r \log x + \{ 2r - 1 + \alpha \} x^r = 0
\] (3.9)

Case 3: \( \Delta = (\alpha - 1)^2 - 4\beta < 0. \)

\[
r_{\pm} = \frac{(1 - \alpha) \pm i \sqrt{4\beta - (\alpha - 1)^2}}{2} = \lambda \pm i\mu
\]

\[ y(x) = c_1x^{\lambda + i\mu} + c_2x^{\lambda - i\mu} \quad x^r = e^{r \ln x}
\]
\[ = c_1e^{(\lambda + i\mu) \ln x} + c_2e^{(\lambda - i\mu) \ln x}
\]
\[ = x^\lambda \{ c_1e^{i\mu \ln x} + c_2e^{-i\mu \ln x} \}
\]
\[ = A_1x^\lambda \cos(\mu \ln x) + A_2x^\lambda \sin(\mu \ln x)
\] (3.10)

Observations:

- If \( x < 0 \) replace by \(|x|\).

The two solutions are linearly independent as we can verify by applying the Wronskian test, as follows:

\[ w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y'_1y_2 \] (look up the definition of the Wronskian)

\[ = \{ x^\lambda \cos(\mu \ln x) \} \{ \ln x x^\lambda \sin(\mu \ln x) + x^{\lambda-1} \cos(\mu \ln x) \mu \} 
- \{ x^\lambda \ln x \cos(\mu \ln x) - x^{\lambda-1} \sin(\mu \ln x) \mu \} \{ x^\lambda \sin(\mu \ln x) \} \]

\[ = \mu x^{2\lambda-1} \text{ independent for } x \neq 0. \] (3.11)

Example 7:

\[ x^2 y'' - xy' - 2y = 0, \quad y(1) = 0, \quad y'(1) = 1 \]
\[ y = x^r \quad r(r - 1) - r - 2 = 0 \quad r^2 - 2r - 2 = 0 \]
\[ (r - 1)^2 = 3 \quad r = 1 \pm \sqrt{3} \]
\[ y = c_1 x^1 + c_2 x^{1-\sqrt{3}} \]
\[ y(1) = c_1 + c_2 = 0 \quad c_2 = -c_1 \]
\[ y(x) = c_1 \left( x^{1+\sqrt{3}} - x^{1-\sqrt{3}} \right) \] (3.12)
\[ y'(x) = c_1 \left[ (1 + \sqrt{3}) x^{\sqrt{3}} - (1 - \sqrt{3}) x^{-\sqrt{3}} \right] \bigg|_{x=1} = c_1 2\sqrt{3} = 1 \]

Therefore
\[ y(x) = \frac{1}{2\sqrt{3}} \left( x^{1+\sqrt{3}} - x^{1-\sqrt{3}} \right). \] (3.13)

Example 8:

\[ x^2 y'' - 3xy' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 0 \]
\[ y = x^r \implies r(r - 1) - 3r + 4 = r^2 - 4r + 4 = 0 \quad (r - 2)^2 = 0 \] (3.14)
\[ y(x) = c_1 x^2 + c_2 x^2 \log x \]
\[ y(1) = c_1 = 1 \quad y'(1) = [2x + c_2 (2x \log x + x)]_{x=1} \]
\[ = 2 + c_2 = 0 \]

Therefore
\[ y(x) = x^2 - 2x^2 \log x. \] (3.15)