Lecture 1: Review of methods to solve Ordinary Differential Equations

(Compiled 3 September 2014)

In this lecture we will briefly review some of the techniques for solving First Order ODE and Second Order Linear ODE, including Cauchy-Euler/Equidimensional Equations.

**Key Concepts:** First order ODEs: Separable and Linear equations; Second Order Linear ODEs: Constant Coefficient Linear ODE, Cauchy-Euler/Equidimensional Equations.

1 First Order ordinary Differential Equations:

1.1 Separable Equations:

\[ \frac{dy}{dx} = P(x)Q(y) \]

\[ \int \frac{dy}{Q(y)} = \int P(x) \, dx + C \]

Example 1:

\[ \frac{dy}{dx} = \frac{4y}{x(y-3)} \]

\[ (\frac{y-3}{y}) \, dy = \frac{4}{x} \, dx \]

\[ y - 3 \ln |y| = 4 \ln |x| + C \]

\[ y = \ln(x^4y^3) + C \]

\[ Ax^4y^3 = e^y \]

1.2 Linear First Order equations - The Integrating Factor:

\[ y'(x) + P(x)y = Q(x) \]

Can we find a function \( F(x) \) to multiply (4.3) by in order to turn the left hand side into a derivative of a product:

\[ Fy' + FPy = FQ \]

\[ (Fy)' = Fy' + F'y = FQ \]
So let \( F' = FP \) which is a separable Eq.

\[
\frac{dF}{F(x)} = P(x) \, dx \Rightarrow \int \frac{dF}{F} = \int P(x) \, dx + C
\]

Therefore \( \ln F = \int P(x) \, dx + C \) \hspace{1cm} (1.6)

or \( F = Ae^{\int P(x) \, dx} \) choose \( A = 1 \)

\( F = e^{\int P(x) \, dx} \) integrating factor

Therefore

\[
e^{\int P(x) \, dx} y' + e^{\int P(x) \, dx} P(x) y = e^{\int P(x) \, dx} Q(x)
\]

\[
\left(e^{\int P(x) \, dx} y\right)' = e^{\int P(x) \, dx} Q(x)
\]

\[ y(x) = e^{-\int P(x) \, dx} \left\{ \int e^{\int P(t) \, dt} Q(x) \, dx + C \right\} \hspace{1cm} (1.7)
\]

Example 2:

\[
y' + 2y = 0 \hspace{1cm} (1.8)
\]

\[
F(x) = e^{2x} \Rightarrow e^{2x} y' + e^{2x} 2y = (e^{2x} y)' = 0 \hspace{1cm} e^{2x} y = c
\]

\[ y(x) = Ce^{-2x} \]

Example 3: Solve

\[
\frac{dy}{dx} + \cot(x) y = 5e^{\cos x}, \quad y(\pi/2) = -4
\]

\[
P(x) = \cot x \quad Q(x) = 5e^{\cos x}
\]

\[
F(x) = e^{\int \cot x \, dx} = e^{\ln(\sin x)} = \sin x
\]

Therefore \( \sin(x)y' + \cos(x)y = (\sin(x)y)' = 5e^{\cos x} \sin x \)

\[
\sin(x)y = -5e^{\cos x} + C
\]

\[ y(x) = -\frac{5e^{\cos x} - C}{\sin x} \hspace{1cm} (1.11)
\]

\[-4 = y(\pi/2) = -\frac{5 - C}{1} \Rightarrow C = 1 \]

Therefore \( y(x) = \frac{1 - 5e^{\cos x}}{\sin x} \)

2 Second Order Constant Coefficient Linear Equations:

\[
Ly = ay'' + by' + cy = 0
\]

Guess \( y = e^{rx} \quad y' = re^{rx} \quad y'' = r^2e^{rx} \)

\[
Ly = [ar^2 + br + c]e^{rx} = 0 \text{ provided } [ ] = 0
\]

Indicial Eq.:

\[
g(r) = ar^2 + br + c = 0 \hspace{1cm} r_{1,2} = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a} \hspace{1cm} (2.1)
\]

Case I: \( \Delta = b^2 - 4ac > 0, r_1 \neq r_2, y(x) = c_1e^{r_1x} + c_2e^{r_2x} \) is the general solution.
Case II: $\Delta = 0$, $r_1 = r_2$, repeated roots $Ly = a(r - r_1)^2e^{rx} = 0$. In this case obtain only one solution $y(x) = e^{rx}$.

How do we get a second solution?

**First Method: Perturbation of the double root:** Consider a small perturbation (see figure 1 a) to the double root case, such that $g(r) = a(r - (r_1 - \epsilon))(r - (r_1 + \epsilon)) = a[(r - r_1)^2 - \epsilon^2] \approx a(r - r_1)^2$. In this case the two, very close but distinct, roots of $g(r) = 0$ are given by:

$$r = r_1 + \epsilon \quad \text{and} \quad r = r_1 - \epsilon \tag{2.2}$$

Now since we still have two distinct roots in this perturbed case, the general solution is:

$$y(x) = c_1 e^{(r_1+\epsilon)x} + c_2 e^{(r_1-\epsilon)x} \tag{2.3}$$

Now choosing a special solution by selecting $c_1 = \frac{1}{2\epsilon} = -c_2$, and we obtain a family of solutions that depend on the small parameter $\epsilon$ (see figure 1 b):

$$y(x, \epsilon) = \frac{e^{(r_1+\epsilon)x} - e^{(r_1-\epsilon)x}}{2\epsilon} \approx \left| \frac{\partial}{\partial r} e^{rx} \right|_{r=r_1} \tag{2.4}$$

Now taking the limit as $\epsilon \to 0$ by making use of L’Hospital’s Rule, we obtain the following limiting solution:

$$y(x, \epsilon) = e^{rx} \left( \frac{e^{\epsilon x} - e^{-\epsilon x}}{2\epsilon} \right) \left. \frac{\partial}{\partial r} e^{rx} \right|_{r=r_1} \tag{2.5}$$

**Second Method: taking the derivative with respect to $r$:** From (2.4) and (2.5) we see that the new solution $xe^{rx}$ was obtained by taking the derivative of $y(x, r) = e^{rx}$ with respect to $r$ and then making the substitution $r = r_1$. This
is, in fact, a general procedure that we will use later in the course. To see why this procedure works, let

\[
y(r, x) = e^{rx}
\]

\[
Ly(r, x) = a(r - r_1)^2e^{rx}
\]

\[
L \left[ \frac{\partial y}{\partial r} (r, x) \right]_{r=r_1} = \left[ 2a(r - r_1)e^{rx} + 2a(r - r_1)xe^{rx} \right]_{r=r_1} = 0
\]

Therefore\[
\frac{\partial y}{\partial r} (r, x) \bigg|_{r=r_1} = xe^{rx} \text{ is also a solution.}
\]

Thus, to summarize, the general solution for the case of a double root is:

\[
y(x) = c_1e^{r_1x} + c_2xe^{r_1x}
\]

Case III: Complex Conjugate Roots: \( \Delta = b^2 - 4ac < 0 \)

\[
r_\pm = \frac{b}{2a} \pm i \left[ \frac{4ac - b^2}{2a} \right]^{1/2} = \lambda \pm i\mu
\]

\[
y(x) = c_1e^{(\lambda+i\mu)x} + c_2e^{(\lambda-i\mu)x}
\]

\[
y(x) = e^{\lambda x} \left[ A \cos \mu x + B \sin \mu x \right].
\]

Example 4:

\[
Ly = y'' + y' - 6y = 0
\]

\[
y = e^{rx} (r^2 + r - 6) = (r + 3)(r - 2) = 0
\]

\[
y(x) = c_1e^{-3x} + c_2e^{2x}
\]

Example 5:

\[
Ly = y'' + 6y' + 9y = 0
\]

\[
y = e^{rx} (r + 3)^2 = 0
\]

\[
y(x) = c_1e^{-3x} + c_2xe^{-3x}
\]

Example 6:

\[
Ly = y'' - 4y' + 13y = 0
\]

\[
y = e^{rx} : \quad r^2 - 4r + 13 = 0
\]

\[
r = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i
\]

Therefore \( y(x) = e^{2x} \left[ A \cos 3x + B \sin 3x \right]. \)

3 Cauchy/Euler/Equidimensional Equations:

\[
Ly = x^2y'' + \alpha xy' + \beta y = 0.
\]

Aside: Note if we let \( t = \ln x \) or \( x = e^t \) then \( \frac{d}{dx} = \frac{d}{dt} \frac{dx}{dt} = \frac{d^2}{dt^2} \). \( \frac{d^2}{dx^2} = x \frac{d}{dx} \left( x \frac{d}{dx} \right) = x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} \Rightarrow x^2 \frac{d^2}{dx^2} = \frac{d^2}{dt^2} - \frac{d}{dt} \).

\[
\frac{d^2}{dx^2} = x \frac{d}{dx} \left( x \frac{d}{dx} \right) = x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} \Rightarrow x^2 \frac{d^2}{dx^2} = \frac{d^2}{dt^2} - \frac{d}{dt}
\]

\[
\therefore \quad \ddot{y} - \dot{y} + \alpha \dot{y} + \beta y = 0
\]

\[
\ddot{y} + (\alpha - 1)\dot{y} + \beta y = 0
\]

\[
y = e^{rt} \Rightarrow r^2 + (\alpha - 1)r + \beta = 0 \quad \text{Characteristic Eq.}
Review of methods to solve Ordinary Differential Equations

Back to (3.1): Guess \( y = x^r, \ y' = r x^{r-1}, \) and \( y'' = r(r-1)x^{r-2}. \)

Therefore \( \{ r(r-1) + ar + \beta \} x^r = 0 \)
\( f(r) = r^2 + (a-1)r + \beta = 0 \) as above.

\[ r_{\pm} = \frac{1 - \alpha \pm \sqrt{(\alpha - 1)^2 - 4\beta}}{2} \] (3.5)

Case 1: \( \Delta = (\alpha - 1)^2 - 4\beta > 0 \) Two Distinct Real Roots \( r_1, \ r_2. \)

\[ y = c_1 x^{r_1} + c_2 x^{r_2} \] (3.6)

If \( r_1 \) or \( r_2 < 0 \) then \( |y| \to \infty \) as \( x \to 0. \)

Case 2: \( \Delta = 0 \) Double Root \( (r - r_1)^2 = 0. \)

We obtain only one solution in this case:

\[ y = c_1 x^{r_1} \] (3.7)

To get a second solution we use second method introduced above, in which we differentiate with respect to the parameter \( r: \)

\[ \frac{\partial}{\partial r} L[x^r] = L \left[ \frac{\partial}{\partial r} x^r \right] = L[x^r \log x] \] (3.8)

\[ \frac{\partial}{\partial r} \{ f(r)x^r \} = f'(r)x^r + f(r)x^r \log x = 0 \quad \text{since} \quad f(r) = (r - r_1)^2. \]

General Solution: \( y(x) = (c_1 + c_2 \log x)x^{r_1}. \)

Check:

\[ L(x^r \log x) = x^2 (x^r \log x)'' + \alpha x (x^r \log x)' + \beta (x^r \log x) - \]
\[ = x^2 \left[ r(r-1)x^r \log x + r x^{r-2} + (r-1)x^{r-2} \right] \]
\[ + \alpha x \left[ r x^{r-1} \log x + x^{r-1} \right] + \beta (x^r \log x) \]
\[ = \{ r^2 + (\alpha - 1)r + \beta \} x^r \log x + \{ 2r - 1 + \alpha \} x^r = 0 \] (3.9)

Case 3: \( \Delta = (\alpha - 1)^2 - 4\beta < 0. \)

\[ r_{\pm} = \frac{(1 - \alpha)}{2} \pm \frac{i \sqrt{4\beta - (\alpha - 1)^2}}{2} = \lambda \pm i\mu \]

\( y(x) = c_1 e^{(\lambda + i\mu) \ln x} + c_2 e^{(\lambda - i\mu) \ln x} \]
\[ = c_1 e^{(\lambda + i\mu) \ln x} + c_2 e^{(\lambda - i\mu) \ln x} \]
\[ = x^\lambda \{ c_1 e^{i\mu \ln x} + c_2 e^{-i\mu \ln x} \} \]
\[ = A_1 x^\lambda \cos(\mu \ln x) + A_2 x^\lambda \sin(\mu \ln x) \] (3.10)

**Observations:**

- If \( x < 0 \) replace by \( |x|\).
The two solutions are linearly independent as we can verify by applying the Wronskian test, as follows:

\[ w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \] (look up the definition of the Wronskian)

\[ = \{ x^\lambda \cos(\mu \ln x) \} \{ \log x x^\lambda \sin(\mu \ln x) + x^{\lambda-1} \cos(\mu \ln x) \mu \}
- \{ x^\lambda \log x \cos(\mu \ln x) - x^{\lambda-1} \sin(\mu \ln x) \mu \} \{ x^\lambda \sin(\mu \ln x) \}
\]

\[ = \mu x^{2\lambda-1} \] independent for \( x \neq 0 \).

Example 7:

\[ x^2 y'' - x y' - 2y = 0, \quad y(1) = 0, \quad y'(1) = 1 \]
\[ y = x^r \quad r(r - 1) - r - 2 = 0 \quad r^2 - 2r - 2 = 0 \]
\[ (r - 1)^2 = 3 \quad r = 1 \pm \sqrt{3} \]
\[ y = c_1 x^{1+\sqrt{3}} + c_2 x^{1-\sqrt{3}} \]
\[ y(1) = c_1 + c_2 = 0 \quad c_2 = -c_1 \]
\[ y(x) = c_1 \left( x^{1+\sqrt{3}} - x^{1-\sqrt{3}} \right) \] (3.12)
\[ y'(x) = c_1 \left[ (1 + \sqrt{3})x^{\sqrt{3}} - (1 - \sqrt{3})x^{-\sqrt{3}} \right] \bigg|_{x=1} = c_1 2\sqrt{3} = 1 \]
Therefore \( y(x) = \frac{1}{2\sqrt{3}} \left( x^{1+\sqrt{3}} - x^{1-\sqrt{3}} \right) \). (3.13)

Example 8:

\[ x^2 y'' - 3xy' + 4y = 0 \quad y(1) = 1 \quad y'(1) = 0 \]
\[ y = x^r \quad \Rightarrow \quad r(r - 1) - 3r + 4 = r^2 - 4r + 4 = 0 \quad (r - 2)^2 = 0 \] (3.14)
\[ y(x) = c_1 x^2 + c_2 x^2 \log x \]
\[ y(1) = c_1 = 1 \quad y'(x) = 2x + c_2 \left[ 2x \log x + x \right]_{x=1} \]
\[ = 2 + c_2 = 0 \]
Therefore \( y(x) = x^2 - 2x^2 \log x \). (3.15)