Let's generalize the notion of vectors:
- a set V of vectors (\( \mathbb{R}^n \))
- add elements of V (standard vector addition in \( \mathbb{R}^n \))
- a set of scalars \( F \) (for \( V = \mathbb{R}^n \), \( F = \mathbb{R} \))
- need addition and multiplication of scalars (usual addition & multiplication in \( \mathbb{R} \))
- need to be able to multiply scalars and vectors (in \( \mathbb{R}^n \), if \( \mathbf{v} \in \mathbb{R}^n \), \( s \in \mathbb{R} \), then \( s\mathbf{v} \) is well-defined!)

What is a vector?

So far:
- an n-tuple \( [x_1, x_2, \ldots, x_n] \) is a vector \( (x_1, x_2, \ldots, x_n \in \mathbb{R}) \) in \( \mathbb{R}^n \)
- a vector is a mathematical object with a magnitude and direction
- we always say that \( \mathbb{R}^n \) is "n-dimensional".
Let's formalize: We say that $V$ is a vector space over the scalars $F$ if:

(1) $(V, +)$ is a commutative group:
   That is, $\forall u, v, w \in V$
   (i) $u + v \in V$
   (ii) $u + v = v + u$
   (iii) $u + (v + w) = (u + v) + w$
   (iv) $\exists \overline{0}$ s.t. $u + \overline{0} = u$
   (v) $\exists -u \in V$ s.t. $u + (-u) = \overline{0}$

(2) For scalars $a, b$ in $F$,
   (vi) $a u \in V$
   (vii) $a(u + v) = au + av$
   (viii) $(a+b)u = au + bu$
   (ix) $1 \cdot u = u$

Examples
(1) $\mathbb{R}^n$ with $V = \mathbb{R}^n$, $F = \mathbb{R}$
(2) $V = \text{set of all } n \times n \text{ matrices}$
   $F = \mathbb{R}$
(3) Let $V = \text{set of all } \theta \text{ functions}$
   on $[0, 1]$.
   $F = \mathbb{R}$
   For ex., $f(x) = \sin x$ on $[0, 1]$ is a "vector" in this vector space.
(4) $V = \text{set of all } \theta \text{ functions}$
   on $(0, 1)$.

Some examples of sets that are not vector spaces:
(1) $V = [0, 1]$. not closed under addition, e.g.
   $0.75 + 0.75 = 1.5 \not\in V$
(2) \( V = [0, \infty) \) is not a vector space (with \( F = \mathbb{R} \)) as additive inverses are not in \( V \).

**Vector subspaces**

Let \( V \) be a vector space. A subset \( S' \) of \( V \) is a **Subspace** of \( V \) if for all \( u, v \in S' \) and for all scalars \( a, b \) we have:

\[
\begin{align*}
(1) & \quad u + v \in S' \\
(2) & \quad au \in S'
\end{align*}
\]

Note: A subspace \( S' \) of \( V \) is itself a vector space.

**Ex:**

(1) Is \( S = \{ \begin{bmatrix} x \end{bmatrix} : x_1 \geq 0 \} \) a vector space?

**Sol:** \( V = \mathbb{R}^3 \); \( S \subseteq V \). So, we need to check if \( S \) is a subspace: take \( u, v \) in \( S \), say \( u = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \); \( v = \begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix} \).

Take \( a, b \in \mathbb{R} \). Then

\[
au + bv = \begin{bmatrix} ax_1 + bx_2 \\ 0 \\ 0 \end{bmatrix} \in S'
\]

\( \Rightarrow \) **YES!**