Recurrences (ctd)

A general Example:

\[ X_{n+1} = 3X_n + X_{n-1} + 2X_{n-2} \]

\[ \begin{bmatrix} X_{n+1} \\ X_n \\ X_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_n \\ X_{n-1} \\ X_{n-2} \end{bmatrix}; \]

if given, include the initial condition: for example, if

\[ X_0 = a; \quad X_1 = b; \quad X_2 = c, \quad \text{then} \]

\[ \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = x^{(0)} \]

Then: \[ x^{(1)} = Ax^{(0)}; \quad x^{(2)} = Ax^{(1)} = A^2x^{(0)} \]

\[ x = \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = A^k \begin{bmatrix} a \\ b \end{bmatrix} \]

\[ x^{(k)} = \begin{bmatrix} x_{k+2} \\ x_{k+1} \\ x_k \end{bmatrix} = A^k \begin{bmatrix} c \\ b \\ a \end{bmatrix} \]

To understand the behaviour, we analyze eigenvalues of \( A \).
Markov Chains

That is:
\[
\sum_{i=1}^{3} P_{ij} = 1 \quad \forall j.
\]

Def.:
\( X_{ni} \): probability of being at node \( i \) after \( n \) weeks

In our example,
\[
X_n = [X_{n1}, X_{n2}, X_{n3}]^T
\]

Note: \( X_{ni} \geq 0 \), \( 0 \leq X_{ni} \leq 1 \) and \( \sum_{i=1}^{3} X_{ni} = 1 \)

Such vectors, we will call state vectors.

* \( P_{ij} \) are "probabilities", so

\[
0 \leq P_{ij} \leq 1
\]

* \( P_{11} + P_{21} + P_{31} = 1 \)

\( P_{12} + P_{22} + P_{32} = 1 \)

\( P_{13} + P_{23} + P_{33} = 1 \)
Suppose we have $x_n = [x_{n1}, \ldots, x_{n3}]$.

We want to compute $x_{n+1}$.

$x_{n+1,i} = x_{n1} \cdot P_{i1} + x_{n2} \cdot P_{i2} + x_{n3} \cdot P_{i3}$

$= [P_{i1} \ P_{i2} \ P_{i3}] [x_{n1} \ x_{n2} \ x_{n3}]$

$X_{n+1} = \begin{bmatrix} x_{n+1,1} \\ x_{n+1,2} \\ x_{n+1,3} \end{bmatrix}$

$P$

We get:

$X_{n+1} = P \cdot x_n$.

↑ a recursion-like expression

If the initial probabilities are $x_0$, then

$X_n = P^n \cdot x_0$.

Now, for a system with $k$ nodes, we get

$P = \begin{bmatrix} P_{i1} \ P_{i2} \cdots \ P_{ik} \\ P_{j1} \ P_{j2} \cdots \\ \vdots \end{bmatrix}$. 
where

(1) $0 \leq P_{ij} \leq 1$

(2) Each column of sums up to 1, i.e.,

$$\sum_{i=1}^{k} P_{ij} = 1 \forall j.$$ 

Such matrices (that satisfy (1) and (2) above) are called **stochastic matrices**.

Back to our problem (with $k=3$)

Given an initial choice of student club, where will the student most likely stay eventually? In other words, we seek the steady-state (or "large n") behavior of this system.

Ex:

$$P = \begin{bmatrix} \frac{3}{4} & 0 & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Suppose the student starts at the std. club #1: $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

After $n=10$ weeks:

$$P^{10} x_0 = \begin{bmatrix} 0.06 & 0.47 & 0.47 \end{bmatrix}$$
For $n = 100$, 

$P^{100} x_0 = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}$