Observe: \[ \frac{\phi^n}{\sqrt{5}} \] is very close to an integer when \( n \) is large.

\[ \phi = \frac{1 + \sqrt{5}}{2} \]

\[ \frac{\phi^n}{\sqrt{5}} = 75.024.999997... \]

Why? Consider the Fibonacci sequence:

\[ 0, 1, 1, 2, 3, 5, 8, 13, 21, ... \]

(\[ F_{n+1} = F_n + F_{n-1}; \ F_0 = 0; \ F_1 = 1 \] (\( \star \))

a recursion whose solution is the Fibonacci sequence.

Rewrite using matrix notation:

\[ \begin{cases} [F_{n+1}] = [1 \ 1] [F_n] \\ [F_n] = [1 \ 0] [F_{n-1}] \end{cases} \]

Note: (\( \star \)) and (\( \star \star \)) are equiv.

Let's focus on (\( \star \star \)):

Want a formula for \( F_n \).

\[
\begin{align*}
[F_2] &= [1 \ 0] [F_1] \\
[F_3] &= [1 \ 0] [F_2] \\
&= [1 \ 0] \cdot [1 \ 0] [F_1]
\end{align*}
\]
Then:
\[
\begin{bmatrix}
F_{n+1} \\
F_n
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\
F_0
\end{bmatrix}
\]

Note: \(
\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}
\) is a real symmetric matrix; thus it is diagonalizable.

Let's find the eigenvalues/eigenvectors of \( A \).

We get: \( \lambda_1 = \frac{1 + \sqrt{5}}{2} \)
\( \lambda_2 = \frac{1 - \sqrt{5}}{2} \)

\[
V_1 = \begin{bmatrix} \lambda_1 \\
1
\end{bmatrix}; \quad V_2 = \begin{bmatrix} \lambda_2 \\
1
\end{bmatrix}
\]

\( A = SDS^{-1} \) where
\[
S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}; \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
\]

\[
S^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}. \quad \text{Then:}
\]

\[
A^n = S \cdot D^n \cdot S^{-1}. \quad \text{Then}
\]

\[
\begin{bmatrix} F_{n+1} \\
F_n
\end{bmatrix} = A^n \cdot \begin{bmatrix} F_1 \\
F_0
\end{bmatrix} = A^n \cdot \begin{bmatrix} 1 \\
0
\end{bmatrix}
\]

\[
= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\
0
\end{bmatrix}
\]

\[
= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 \\
-\lambda_2
\end{bmatrix}
\]

\[
= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix}
\]
\[
\begin{bmatrix}
F_{n+1} \\
F_n
\end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix}
\lambda_1 & \lambda_2 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
\lambda_1^n \\
\lambda_2^n
\end{bmatrix}
\]

Then:
\[
F_n = \frac{1}{\sqrt{5}} (\lambda_1^n - \lambda_2^n)
\]

\[
= \frac{\lambda_1^n}{\sqrt{5}} \left(1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n\right)
\]

noting that \(\left|\frac{\lambda_2}{\lambda_1}\right| < 1\), we get
\[
limit_{n \to \infty} \left(\frac{\lambda_2}{\lambda_1}\right)^n = 0
\]
and so:
for large \(n\),
\[
F_n \approx \frac{\lambda_1^n}{\sqrt{5}} = (\phi + \sqrt{5}/2)^n \cdot \frac{1}{\sqrt{5}}
\]

Since \(F_n\) is an integer \(\text{at is the } n\text{th entry of the Fibonacci sequence, this explains why}
\[
\frac{\lambda_1^n}{\sqrt{5}}
\]

is very close to an integer when \(n\) is large.