Recall: Let \( \{ q_k \} \) be an ON basis for \( V \). Then for any \( v \in V \), we have
\[
\begin{align*}
\overrightarrow{v} &= \langle q_1, v \rangle q_1 + \langle q_2, v \rangle q_2 \\
&\quad + \cdots + \langle q_n, v \rangle q_n \\
&= \sum_{k=1}^{n} \langle q_k, v \rangle q_k
\end{align*}
\]

We can also use matrix notation:
\[
Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}
\]

Then
\[
v = Q \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}
\]

where
\[
\begin{align*}
Q^* v &= \overrightarrow{Q^* v} \\
&= \langle v, Q^* v \rangle
\end{align*}
\]

Fact: If \( \{ q_k \} \) is an ON basis for \( V \), then
\[
\sum_{k=1}^{n} |c_k|^2 = \| v \|_2^2
\]

where \( c_k = \langle q_k, v \rangle \).

Proof:
\[
v = \sum_{k=1}^{n} c_k q_k
\]

\[
\| v \|_2^2 = \langle v, v \rangle = \langle \sum_{k=1}^{n} c_k q_k, \sum_{k=1}^{n} c_k q_k \rangle
\]

\[
= \sum_{k=1}^{n} \sum_{j=1}^{n} c_k \overline{c_j} \langle q_k, q_j \rangle
\]

\[
= \sum_{k=1}^{n} |c_k|^2
\]
So:
\[ \|v\|^2 = |c_1|^2 + |c_2|^2 + \ldots + |c_n|^2 \]
as claimed.

Remark: Using (\(\times\)), we can rewrite this as
\[ \|Q^*v\| = \|v\| \]

Orthogonal & unitary matrices

If we make a matrix \(Q\) with columns \(Q_i\) consisting of vectors in an ON basis (so \(Q\) must be square), then we call \(Q\)

(i) Orthogonal (if all entries are real)

(ii) Unitary (if entries are complex).

Basic properties

(1) If \(Q\) is unitary, then
\[ Q^*Q = I_n \]
\[ \Rightarrow Q^* = Q^{-1} \] (inverse)

Why?
\[
\begin{bmatrix}
q_1^* \\
q_2^* \\
\vdots \\
q_n^*
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
\vdots \\
q_n
\end{bmatrix}
= 
\begin{bmatrix}
Q_{ij}
\end{bmatrix}
\]
\[ Q_{ij} = \langle q_i, q_j \rangle = \delta_{ij} \]

Conjugate transpose
where \( \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \)

(Note: if \( Q \) is orthogonal, \( Q^{T}Q = I \))

A corollary:
We also have \( QQ^{*} = I \)
because \( Q^{*} = Q^{-1} \).

Then, if the columns of a matrix \( Q \) form an ON basis, then so does its rows.

(2) The columns of any orthonormal basis are linearly independent.

(3) A matrix \( Q \) is unitary iff \( ||Qv|| = ||v|| \) for all \( v \).

(Pf: see online notes)
Ch. 4 Eigenvalues & Eigenvectors

**Def:** Let $A$ be an $n \times n$ (square) matrix. A number $\lambda$ (in $\mathbb{R}$ or $\mathbb{C}$) and a non-zero vector $\vec{v}$ (in $\mathbb{R}^n$ or in $\mathbb{C}^n$) are an eigenvalue-eigenvector pair if

$$A\vec{v} = \lambda \vec{v} \quad (\ast)$$

**Remarks:**

1. $\vec{v} \neq \vec{0}$, but $\lambda$ can be 0.
2. Note that $(\ast)$ is equivalent:

$$A\vec{v} = \lambda \vec{v} \iff A\vec{v} = \lambda I \vec{v} \iff (\lambda I - A)\vec{v} = \vec{0}$$

$(\Rightarrow) \vec{v}$ is in $N(\lambda I - A)$.

Noting that $\lambda I - A$ is an $n \times n$ matrix, there is a non-zero vector in its nullspace iff

$$\det(\lambda I - A) = 0$$

To find the eigenvalues/eigenvectors of a matrix $A$:

1. Find $\lambda$ for which $\det(\lambda I - A) = 0$ → Eigenval.
2. For each eigenvalue $\lambda$, solve the homog. system $(\lambda I - A)\vec{v} = \vec{0}$, for $\vec{v} \rightarrow$ eigenvectors.
Ex: Find the eigenvalues/eigenvectors of
\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \].

\[ \text{Sol: } \det(\lambda I - A) = 0 \]
\[ = \det \begin{bmatrix} \lambda & 0 \\ 0 & \lambda - 2 \end{bmatrix} = 0 \]
\[ \Rightarrow (\lambda - 1)(\lambda - 2) = 0 \]
\[ \Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = 2 \text{ are the eigenvalues.} \]

\[ \text{Corresp. eigenvectors} \]
\[ \lambda_1 = 1 \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} v_1 \end{bmatrix} = 0 \]
\[ \Rightarrow v_1 = t; \quad v_2 = 0 \]
\[ \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ is an eigenvector for } \lambda_1 = 1. \]

\[ \lambda_2 = 2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \end{bmatrix} = 0 \]
\[ \Rightarrow v' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ is an eigenvector for } \lambda_2 = 2. \]