Complex vector spaces

\[ \mathbb{C}^n = \{ z = [z_1 \ldots z_n] : z_j \in \mathbb{C} \} \]

- All basic properties of \( \mathbb{R}^n \) generalize to \( \mathbb{C}^n \) after replacing real scalars with complex scalars.

\[ z = [z_1 \ldots z_n] ; w = [w_1 \ldots w_n] ; \text{se} \mathbb{C} \]

\[ z + w = [z_1 + w_1 \ldots z_n + w_n] \]

\[ \bar{z} = [\bar{z}_1 \ldots \bar{z}_n] \]

Define \( \bar{z} = [\bar{z}_1 \ldots \bar{z}_n] \), the complex conjugate of the vector \( z \).

Need to change the defn of the inner product: For \( z, w \) in \( \mathbb{C}^n \),

\[ \langle w, z \rangle = \bar{w}_1 z_1 + \ldots + \bar{w}_n z_n \]

Why do we need to do this change?

Recall: For \( x \in \mathbb{R}^n \), \( \langle x, x \rangle = \|x\|^2 \)

Want "this" to hold for \( z \in \mathbb{C}^n \):

\[ \langle \bar{z}, z \rangle = \bar{z}_1 z_1 + \bar{z}_2 z_2 + \ldots + \bar{z}_n z_n = \|z\|^2 \]

\[ = 1\|z_1\|^2 + 1\|z_2\|^2 + \ldots + 1\|z_n\|^2 \]

\[ = \|z\|^2 \]
Some consequences

1. For $s \in \mathbb{C}$, $z, w \in \mathbb{C}^n$,
   $\langle sz, z \rangle = s^2 \langle w, z \rangle$
   $\langle w, sz \rangle = s \langle w, z \rangle$

2. $\langle w, z \rangle = \langle z, w \rangle$

3. $\langle w, z \rangle = \overline{w}^T z$
   $= [\overline{w}_1 \ldots \overline{w}_n] [\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}]$

4. $\|z\|^2 = \sum_{j=1}^{n} |z_j|^2$

5. Recall: for $x, y \in \mathbb{R}^n$,
   \[ \langle x, Ay \rangle = \langle A^T x, y \rangle. \]
   What about $z, w \in \mathbb{C}^n$,
   $A$ a complex matrix
   (i.e. $A = [a_{ij}]$; $a_{ij} \in \mathbb{C}$)
   \[ \langle w, Az \rangle = (\overline{w}^T A) z \]
   \[ = (A^T \overline{w})^T z \]
   \[ = (A^T \overline{w})^T z \]
   \[ = \langle \overline{A}^T w, z \rangle \]

Define: $A^* := \overline{A}^T$ (adjoint of $A$)
Then, we have
\[ \langle w, A z \rangle = \langle A^* w, z \rangle \]
for all \( w, z, A \) compatible.

**Example:**
\[ A = \begin{bmatrix} 1 & 1+2i & 3 \\ i & 2 & 1 \end{bmatrix} \]

For every \( w \in \mathbb{C}^2, z \in \mathbb{C}^3 \), we have
\[ \langle w, A z \rangle = \langle A^* w, z \rangle, \]
where
\[ A^* = \begin{bmatrix} 1 & -i \\ 1-2i & 2 \\ 3 & 1 \end{bmatrix} \]

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**MATLAB:**
- \( i = \text{sqrt}(-1) \)
- Matlab handles complex numbers seamlessly.
- \( A^\dagger = \text{adjoint (i.e. } A^\top) \)
- \( A^\dagger = \text{transpose (} A^\top) \)

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**Orthonormal basis**

Recall: Given a vector space \( V \), \( \{e_1, \ldots, e_n\} \) is a basis if:
- \( x \) spans \( \{e_i\} \) is \( V \), and
- \( \{e_1, \ldots, e_n\} \) is linearly independent.
Ex: \{ (1), (1') \} and \\
\{ (0), (1) \} are both bases for \( \mathbb{R}^2 \).

Def: A basis \( \{ q_1, q_2, \ldots, q_n \} \) is an orthonormal basis for \( V \) if
* \( \langle q_i, q_j \rangle = 0 \) for \( i \neq j \)
* \( \langle q_i, q_i \rangle = 1 \) for all \( i \)

Ex: \( \{ (1), (1') \} \) is not orthonormal.
* \( \{ (1), (0) \} \) is an orthonormal basis for \( \mathbb{R}^2 \).
* \( \{ (1/\sqrt{2}), (1/\sqrt{2}) \} \) is an orthonormal basis for \( \mathbb{R}^2 \).

To verify:
* \( \langle (1/\sqrt{2}), (-1/\sqrt{2}) \rangle = 0 \)
* \( \| (1/\sqrt{2}) \| = 1 \).

Ex: \{ e_1, \ldots, e_n \} is the standard basis of \( \mathbb{R}^n \).
Let \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \) be an orthonormal basis for \( V \).

On the other hand, \( \mathbf{v} \) is a basis for \( \mathbb{R}^n \). Thus, any orthonormal basis for \( \mathbb{R}^n \) is also an orthonormal basis for \( V \).

Orthogonal basis for \( \mathbb{R}^n \) is not true.

\[ \{ \mathbf{v}_1, \mathbf{v}_2 \} \] is an orthogonal basis for \( \mathbb{R}^2 \), but not for \( \mathbb{R}^3 \). On the other hand, \( \mathbf{v} \) is not a basis for \( \mathbb{R}^3 \).

Remark: Recall that all vectors in \( \mathbb{R}^n \) are also in \( V \).

However, since \( \mathbf{v} \) is orthonormal, we can solve for each \( c_i \) as follows.

\[ \mathbf{a} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_n \mathbf{v}_n \]

In general, to find \( c_i \), we need to solve:

\[ \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{a} \]

Let \( \mathbf{v} \in V \) be arbitrary.