What remains is \((C2): \quad P_i'(x_{i+1}) = P_{i+1}'(x_{i+1})\)

... explicit calculation

\[
\frac{(x_{i+1} - x_i)}{6} z_i + \frac{(x_{i+2} - x_i)}{3} z_{i+1} + \frac{(x_{i+2} - x_{i+1})}{6} z_{i+2} = \frac{y_{i+2} - y_i}{x_{i+2} - x_{i+1}} - \frac{y_{i+1} - y_i}{x_{i+1} - x_i}
\]

for \(i = 1, 2, \ldots, n-1\)

Let's put together all equations in the form of a linear system: \(Sz = b\)

\[
b = \begin{bmatrix} 0, \frac{y_3 - y_2}{x_3 - x_2}, \frac{y_2 - y_1}{x_2 - x_1}, \ldots, \frac{y_{n-1} - y_{n-2}}{x_{n-1} - x_{n-2}}, \ldots \end{bmatrix}^T
\]

Then \(z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = S^{-1}b\)

\(S\), obtained this way, is well-conditioned and structured (tri-diagonal) which allows fast computation!
I.3 Finite Difference Approx.

Physical systems (that are inherently continuous) are often described by differential equations. 

**Example:**

\[ f''(x) + q(x)f(x) = r(x) \quad (**) \]

(where \( q(x) \) & \( r(x) \) are given functions and we want to find \( f(x) \).) 

**Specific** \( f''(x) = 4 \quad (***) \)

**Solution:**

\[ f'(x) = 4x + c_1 \]

\[ f(x) = 2x^2 + c_1x + c_2 \]

Solves (***) 

Here \( c_1, c_2 \) are arbitrary constants!

To get a problem with a unique solution, we need additional constraints:

---

Possibility 1: "initial values" of \( f \) and \( f' \)

i.e., \( f(0) = 1 \) & \( f'(0) = 2 \)

(That is, the problem now is:

\[ f''(x) = 4 \] \quad initial value

\[ f(0) = 1 \] & \( f'(0) = 2 \] \quad Problem (IVP)

Possibility 2: "boundary values" of \( f \)

i.e., \( f(0) = 1 \) and \( f(1) = 6 \)

\[ f''(x) = 4 \] \quad boundary value

\[ f(0) = 1 \] & \( f(1) = 6 \] \quad Problem (BVP)

To solve:

Step 1: Find the "general solution:

\[ f(x) = 2x^2 + c_1x + c_2 \]

Step 2: \( f(0) = c_2 = 1 \) & \( f(1) = 2 + c_1 + c_2 = 6 \)

\[ \Rightarrow c_1 = 3 \]

\[ f(x) = 2x^2 + 3x + 1 \]
Problem: Often, even simple looking diff. eqns are hard to solve explicitly - OR there is NO explicit closed-form solution!

\[ f''(x) + \cos(x) f(x) = x^2; \quad f(0) = f(1) = 1 \]

Remedy: Find an approximate (numerical) solution - by discretizing the BVP and turning it into a matrix equation.

\[
\begin{align*}
F &= \begin{bmatrix} f_0 & f_1 & \ldots & f_N \end{bmatrix} \quad \text{discrete N-point approx. of } f(x) \\
\text{(note: } F \in \mathbb{R}^{N+1} \text{)}
\end{align*}
\]

Need: (i) approximate the first derivative \( f'(x) \)

\[
\begin{align*}
\frac{f(x)}{x} &\approx \frac{f_{i+1} - f_i}{x_{i+1} - x_i} = \frac{f_i - f_{i-1}}{\Delta x} \\
\text{so: } f'(x_j) &= \frac{f_{j+1} - f_j}{\Delta x}
\end{align*}
\]

Then: \( F' = (\Delta x)^{-1} \begin{bmatrix} f_1 - f_0 \\ f_2 - f_1 \\ \vdots \\ f_N - f_{N-1} \end{bmatrix} \in \mathbb{R}^N \approx \begin{bmatrix} f'(x_0) \\ f'(x_1) \\ \vdots \\ f'(x_{N-1}) \end{bmatrix}
\]

\( f_j = f(x_j); \quad x_j = x_0 + j \cdot \Delta x \)

\( j = 0, 1, \ldots, N \)