Observe: The matrix \( P \) that we constructed in our last lecture is NOT a proper stochastic matrix. Therefore, we'll modify it to guarantee that such a limit exists and is fast to compute.

Step 1: Replace \( P \) with a proper stochastic matrix \( S \) by:

\[
S = \begin{bmatrix}
0 & \frac{1}{4} & \frac{1}{3} & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} & 1 \\
\frac{1}{4} & 0 & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{3} & 0
\end{bmatrix}
\]

Let's ask again:
- Is there a limit of \( S^n x_0 \)?

Use MATLAB: \( \lambda = 1 \) is the dominant eigenvalue (i.e. \( |\lambda_2| < 1 \) for all other eigenvalues).

To find a limit \( \mathbf{v} \) we can find it using the power method...

MATLAB \( \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.32, 0.81, 0.36, 0.32 \end{bmatrix}^T \)

- eigenvector corresponding to \( \lambda = 1 \).

A better convention: normalize \( \mathbf{v}_1 \) s.t. its entries add up to 1. \( \Rightarrow \)

\[ \mathbf{v}_1 = \begin{bmatrix} 0.18, 0.44, 0.20, 0.18 \end{bmatrix}^T. \]

Paper rank: \( \frac{1}{2} \) rank of 1.
Another example:

\[ S = P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \text{stochastic matrix} \]

\[ S \]

and \[ |\lambda_1| = |\lambda_2| = |\lambda_3| = 1 \]

so, no dominant eigenvalue; thus our method does NOT work.

To resolve: use DAMPING

- Create a matrix \( Q \), same size as \( S \), st. all entries of \( Q \) are identical & \( Q \) is stochastic.

In our case:

\[ Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \]

- Pick a "damping factor" \( \alpha \in [0, 1] \)
- Define the "Google matrix" as

\[ G = \alpha S + (1 - \alpha) Q \]

with damping factor \( \alpha \).

Notes:

1. \( G \) is a stochastic matrix and each entry of \( G \) is positive.
Thus \( \lambda = 1 \) is the dominant eigenvalue of \( G \); corresponding eigenvector has positive entries, giving positive PageRank to every site.
(2) We define the PageRank of a site as the corresponding entry of the eigenvector \( \tilde{v}_i \) of the eigenvector \( \lambda = 1 \), normalize so that its entries add up to 1.

(3) Since all other eigenvalues are strictly less than 1 in magnitude, we can use the power method to compute PageRank.

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**Singular Value Decomposition** (SVD) III. 7.1-3

Idea: Can we generalize "diagonalization" to arbitrary (non-square, ...) matrices?

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Why is this important?

(1) If \( A = S D S^{-1} \) (diag. as before), then
\[
A^k = S D^k S^{-1}, \quad k \geq 0
\]
and also for \( k < 0 \) if
\[
D_{ij} = \lambda_{ij} \neq 0 \quad \forall j.
\]

(2) If \( A \) is unitarily diagonalizable (e.g., when \( A \) is Hermitian), then
\[
\|A_{\text{op}}\| = \max \{ |\lambda_j| \}
\]
where \( \lambda_j \) are the eigenvalues of \( A \).

(3) If \( A \) is diagonalized, then
\[
\text{rank}(A) \neq \# \text{ non-zero eigenvalues}.
\]
SVD: generalize these to arbitrary matrices

The "formula":

\[ A = U \Sigma V^* \]

\( U \) is \( m \times n \); unitary
\( \Sigma \) is \( n \times n \); "diagonal"

\( V \) is \( n \times n \); unitary

Example:

\[
\begin{bmatrix}
1 & 1 \\
1 & -1 \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{6}} \\
-\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

"diagonal" for non-square matrices

How do we find \( U, \Sigma, V \) given \( A \)?

Let \( A \) be \( m \times n \).

Proposition 1: All eig. values of \( A^*A \) are non-negative (and strictly positive if \( A \) is invertible).

Proof: \( A^*A \) is positive semi-definite:

\[ v^*A^*A v = \lambda v^*v \]

\[ \Rightarrow \lambda \geq 0 \]

\[ \langle Av, Av \rangle = \lambda \langle v, v \rangle \geq 0 \]

\[ \lambda \geq 0 \]

Note: if \( A \) is invertible, \( Av \neq \vec{0} \), so \( \lambda > 0 \).