Examples

1. A is n × n; invertible. What is N(A)?
   Sol: Solve
   \[ A^{-1}(A\mathbf{x}) = 0 \]
   \[ \Rightarrow A^{-1}(A\mathbf{x}) = A^{-1}0 \]
   \[ \Rightarrow \mathbf{x} = 0. \]
   \[ \Rightarrow N(A) = \{0\} \]

2. \[ B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \ N(B) = ? \]
   \[ x_3 = s; \quad x_1 = 0; \quad x_2 = 0 \]
   \[ \Rightarrow N(B) = \{ s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; s \in \mathbb{R} \} = \text{Span} \{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \} \]

3. In general, to find N(A), just solve the homog. eqn \[ A\mathbf{x} = 0. \]

Ex.
\[ C = \begin{bmatrix} 1 & 3 & 3 & 10 \\ 2 & 6 & -1 & -1 \\ 1 & 3 & 1 & 4 \end{bmatrix}; \ N(C) = ? \]

Sol:
\[ C \sim \ldots \sim \text{ref}(C) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ x_2 = s; \quad x_4 = t \quad \text{(free variables)} \]
\[ x_1 + 3s + t = 0 \Rightarrow x_1 = -3s - t \]
\[ x_3 + 3t = 0 \Rightarrow x_3 = -3t \]
\[ N(C) = \{ s \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}; s, t \in \mathbb{R} \} = \text{Span} \{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix} \} \]
Notes:

1. \( R(A) \subseteq \mathbb{R}^m \)

2. \( A = [a_1 | a_2 | \ldots | a_n] \)

where \( a_j \in \mathbb{R}^m \) are the columns of \( A \). Then, for \( x \in \mathbb{R}^n \)

\[
A x = [a_1 | a_2 | \ldots | a_n] [x_1 \\
x_2 \\
\vdots \\
x_n] = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n
\]

\( \Rightarrow R(A) = \text{Span}\{a_1, a_2, \ldots, a_n\} \)

= "column space" of \( A \)

Then, to find a basis for \( R(A) \), find the largest set of linearly independent columns of \( A \)

We know how to do this! Find the pivot columns of \( A \)!

Ex: Consider the matrix \( C \) from our previous example. Find a basis for \( R(C) \).

\[
\text{ref}(C) = \begin{bmatrix}
1 & 3 & 0 & 1 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Pivot col. are col #1 & col #3

\( \begin{bmatrix}1 \\
3 \\
-1
\end{bmatrix} \) is a basis for \( R(C) \).

Note: \( \dim (R(C)) = 2 \)
**Def:** \( \text{dim}(\mathbb{R}(A)) \) is called the rank of \( A \), and denoted by \( \text{rank}(A) \).

**Facts:**
1. \( \text{rank}(A) = \# \text{ pivots in } \text{ref}(A) \).
2. Let \( A \) be \( m \times n \). Then
   - \( \text{rank}(A) \leq \min\{m, n\} \)
   - \( \text{dim}(\mathbb{N}(A)) = n - \# \text{ pivots} \)

\[ \Rightarrow \text{dim}(\mathbb{N}(A)) = n - \text{rank}(A) \]

3. Let \( \text{ref}(A) := U \). Then there exists an invertible matrix \( L \) s.t. \( A = LU \).
   
   Then \( A^T = U^T L^T \).

   **Claim:** If \( L \) is invertible, so is \( L^T \).

   **Exercise:** prove this claim.

**Thm:** \( \mathbb{R}(A^T) = \mathbb{R}(U^T) \).

**Remark:** \( [\text{ref}(A)]^T \neq \text{ref}(A^T) \).

**Proof of the Thm:**
\[
\mathbb{R}(A^T) = \{ A^T x : x \in \mathbb{R}^m \} = \{ U^T L x : x \in \mathbb{R}^m \}
\]
Q: Do we have some $\mathbb{R}^m$ such that $L^T x : x \in \mathbb{R}^m$?

i.e., can we find, for every $y \in \mathbb{R}^m$, an $x \in \mathbb{R}^m$ such that $y = L^T x$?

A: Yes. In fact, given $y \in \mathbb{R}^m$, let $x = (L^T)^{-1} y$.

Therefore $L^T x = L^T (L^T)^{-1} y = y$.

Therefore $\mathbb{R}(A^T) = \{ U^T y : y \in \mathbb{R}^m \} = \mathbb{R}(U^T)$.

Ex: Let $C$ be as before. Find $\mathbb{R}(C^T)$.

Sol: $\mathbb{R}(C^T) = \mathbb{R}(U^T)$ where

$$U = \text{ref}(C) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow U^T = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

$$\Rightarrow \mathbb{R}(C^T) = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right\}$$

A basis for $\mathbb{R}(C^T)$.
Ex: Let
\[
A = \begin{bmatrix}
1 & 3 & 4 & 5 & 1 \\
2 & 2 & 2 & 2 & 2 \\
1 & 2 & 4 & 7
\end{bmatrix}
\]
(a) What is \( \text{rank}(A) \)?
(b) \( \dim(N(A)) \)? \( \dim(R(A)) \)?
\( \dim(N(A^T)) \)? \( \dim(R(A^T)) \)?
(c) \( x_4 = s \); \( x_5 = t \) ... exercise.
(d) \( \{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \} \).
(e) \( \text{ker}(A^T) = \text{ker}(A) \cup \text{ker}(A) \).

Sol: First, calculate \( \text{rref}(A) \):
\[
\begin{bmatrix}
1 & 0 & 0 & -2 & -5 \\
0 & 0 & 0 & 5 & 18 \\
0 & 0 & 0 & 0 & 12
\end{bmatrix}
\]
(a) \( \text{rank}(A) = 3 \)

\( \text{ker}(A) = \text{ker}(A) \cup \text{ker}(A) \).
Pf: exercise.