II. 1 Vector Spaces & Subspaces

What is a vector?

- an n-tuple is a vector
  
  \[
  \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
  \end{bmatrix} \in \mathbb{R}^n \quad (x_i \in \mathbb{R})
  \]

  (has a direction & magnitude)

- but also a polynomial is a vector! Or a function on \([0,1]\)!

What do we need?

- a set of vectors \(V (\mathbb{R}^n)\)
- vector addition in \(V\)
  (standard, component-wise addition in \(\mathbb{R}^n\))
- a set of scalars \(F\)
  (for \(V=\mathbb{R}^n, F=\mathbb{R}\))
- addition (subtraction)/multiplication of scalars
  (standard addition/multiplication in \(\mathbb{R}\))
- multiplication of a vector by a scalar
  (in \(\mathbb{R}^n: s \cdot \mathbf{u} = \begin{bmatrix} s u_1 \\ s u_2 \\ \vdots \\ s u_n \end{bmatrix} \))
Formally: We say that $V$ is a vector space over the scalars $F$ if:

(i) $(V, +)$ is a commutative group. That is, $\forall u, v, w \in V$
(ii) $u + v \in V$, (ii) $u + v = v + u$
(iii) $u + (v + w) = (u + v) + w$
(iv) $\exists \overrightarrow{0}$ s.t. $u + \overrightarrow{0} = u \quad \forall u \in V$
(v) $\exists -u \in V$ s.t. $u + (-u) = \overrightarrow{0}$.

(ii) For scalars $a, b \in F$,
(i) $au \in V$
(ii) $a(u + v) = au + av$
(iii) $(a + b)u = au + bu$
(iv) $1 \cdot u = u$
(v) $a(bu) = (ab)u$

Examples

1. $V = \mathbb{R}^n$, $F = \mathbb{R}$
2. $V$: set of all $n \times n$ matrices
   $F = \mathbb{R}$
   with $sM := \left[ sM_{ij} \right]_{n \times n}$
3. $V$: set of all real-valued functions on $[0, 1]$;
   $F = \mathbb{R}$
   For ex., $f(x) = e^x \cos x^2$ is a "vector" with this defn.
4. $V$: set of all cts functions on $\mathbb{R}$, $F = \mathbb{R}$
5. $V = [0, 1]$ is NOT a vector space. See next page.
Why not?

- no additive inverse of any \( x \in [0, 1] \), \( x \neq 0 \)
- not closed under addition:
  \[
  \frac{3}{4} + \frac{3}{4} \notin [0, 1]
  \]
  while \( \frac{3}{4} \in [0, 1] \).

**Vector subspaces**

**Def:** let \( V \) be a vector space (with scalars \( \mathbb{F} \)). A subset \( S \subseteq V \) is a subspace of \( V \) (thus a vector space itself) if:

\[
\forall u, v \in S \quad \text{and} \quad \forall a, b \text{ scalars}
\]

\[
\begin{align*}
\text{(i)} & \quad u + v \in S \\
\text{(ii)} & \quad au \in S
\end{align*}
\]

(\text{or} \ (i')) \quad \text{additive closure}