\[(a_{22}a_{11} - a_{12}a_{21})x_1 = a_{22}b_1 - a_{12}b_2\] 
\[
\frac{a_{22}a_{11} - a_{12}a_{21}}{\det(A)}
\]

**Case 1:** \(a_{22}a_{11} - a_{12}a_{21} \neq 0\) (i.e., \(\det(A) \neq 0\))

\[x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}}\]

Plug into (1):

\[x_2 = \frac{-a_{21}b_1 + a_{11}b_2}{a_{22}a_{11} - a_{12}a_{21}}\]

**Is unique solution!**

**Case 2:** \(a_{22}a_{11} - a_{12}a_{21} = 0\)

(a) If \(a_{22}b_1 - a_{12}b_2 \neq 0\), then **no solution**!

(b) If \(a_{22}b_1 - a_{12}b_2 = 0\), then **infinitely many solutions**!

**Remark:**

1. For an \(n \times n\) system (\(n\) eqns, \(n\) unknowns), we have
2. The system \(Ax = b\) has a unique solution if \(\det(A) \neq 0\).
3. If \(\det(A) = 0\), then the system has either no solution or infinitely many solutions.

4. In general, any linear system will have either a unique solution, or infinitely many solutions, or no solution at all.
Question 2: How do we find the solutions of linear systems?

Problem: Solve

\[ A \cdot x = b \]

when we know \( A, b \).

We want to identify the set of all solutions.

Method: Gaussian elimination

How?

Step 1: Build the augmented matrix

\[ [A : b] \]

Step 2: Using elementary row operations, bring the augmented matrix into (reduced) row echelon form (\( rref \)).

Ex: Solve

\[ \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ A \cdot x = b \]

Solve:

\[ \begin{bmatrix} 3 & 2 & 3 & 0 \\ 1 & -1 & -4 & 1 \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} 0 & 5 & 15 & -3 \\ 1 & -1 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 5 & 15 & -3 \\ 0 & 1 & 3 & -3/5 \end{bmatrix} \]

\[ \overset{\text{Pivots}}{x_1, x_2, x_3} \]

\[ \overset{\text{Reduced row echelon form (rref)}}{\begin{bmatrix} 1 & 0 & -1 & 2/5 \\ 0 & 1 & 3 & -3/5 \end{bmatrix}} \]
The corresponding equations are:
\[ x_1 - x_3 = \frac{2}{5} \quad (1) \]
\[ x_2 + 3x_3 = -\frac{3}{5} \quad (2) \]

- Variables that correspond to non-pivot columns are "free" variables: \( x_3 = t \), \( t \in \mathbb{R} \).
- Then Eqn. (1) \( \Rightarrow \)
  \[ x_1 = t + \frac{2}{5} \]

Eqn. (2) \( \Rightarrow \)
\[ x_2 = -3t - \frac{3}{5} \]

Then the solution set is
\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{2}{5} \\ -\frac{3}{5} \\ 0 \end{bmatrix}, \quad t \in \mathbb{R} \]

A line in \( \mathbb{R}^3 \)

**Special Case:** Suppose \( A \) is a square matrix, say \( n \times n \). Also, suppose \( \det(A) \neq 0 \).

Then the system
\[ Ax = y \]
has a **unique** solution for any \( y \in \mathbb{R}^n \).

Interpret \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) as a function (or a mapping). We then have that \( A \) is one-to-one and onto, thus it has an **inverse** \( A^{-1} \) s.t.
\[ A^{-1} y = x \]

**Fact:** \( A^{-1} \) is also an \( n \times n \) matrix.
Next, define
\[
I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \forall \text{ n x n}
\]

Note: \( I_n \cdot x = x \quad \forall x \in \mathbb{R}^n \)

Def: \( I_n \) is said to be the \text{n-dimensional identity matrix}

\( \uparrow \) Takes on the role of 1 in matrix multiplication!

\[ \begin{align*}
& \cdot A \cdot I_n = A \quad \forall \text{ m x n A} \\
& \cdot I_n B = B \quad \forall \text{ n x m B} \\
& \cdot I_n x = x \quad \forall x \in \mathbb{R}^n.
\end{align*} \]

\[ \text{let's combine: Suppose } A \text{ is n x n with } \det(A) \neq 0. \text{ Then}
\]

(i) \( A^{-1} \) exists and is an n x n matrix. (We say that A is invertible)

(ii) \( AA^{-1} = A^{-1}A = I_n \)

In this case, the solution of \( Ax = b \) is given by
\[ x = A^{-1}(Ax) = A^{-1}b. \]

How do we compute \( A^{-1} \)?

Build \([A : I_n] \rightarrow \text{Gaussian Elim.}\)

\[ \cdots \rightarrow [I_n : A^{-1}] \]
NORMS OF VECTORS & MATRICES

NORMS OF VECTORS (I.15)

Given \( x \in \mathbb{R}^n \), want to find a way of defining the magnitude or size of \( x \).

- For \( n = 1 \), \( x \in \mathbb{R} \), and
  \[ |x| \]
  does the job

- For \( n = 2 \), \( x \in \mathbb{R}^2 \)
  \[ ||x|| = \sqrt{x_1^2 + x_2^2} \]

- General \( n \),
  \[ ||x|| = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2} \]