Ch. 1 Linear Equations

What is a linear eqn?

\[ y = 3x + 2 \] → linear
\[ y = 3x^2 + 2 \] → not linear
\[ y = 3x - 2 = 0 \] → linear
\[ x + y + 2xy = 5 \] → not linear

An equation \( f(x_1, x_2, \ldots, x_n) = b \) is linear iff \( f \) is a polynomial of degree 1.

In general, a linear eqn with \( n \) variables is of the form
\[ a_1x_1 + a_2x_2 + \ldots + a_nx_n = b, \quad (*) \]
\( (a_j, b) \in \mathbb{R} \)

Here: \( x_j \) are variables
\( a_j, b \) are coefficients.

Next, rewrite (*) using matrix notation:
\[
\begin{bmatrix}
  a_1 & a_2 & \cdots & a_n \\
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{bmatrix}
= 
\begin{bmatrix}
  b_1 \\
\end{bmatrix}
\]

Any linear equation with \( n \) variables is of this form.

Next: A bunch of linear equations that have the same solution (i.e., we need to solve them simultaneously) is called a system of linear equations or a linear system.
Ex: \[ 3x_1 + 2x_2 + 3x_3 = 0 \]
\[ x_1 - x_2 - 4x_3 = 1 \]

\[
\begin{bmatrix}
3 & 2 & 3 \\
1 & -1 & -4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

let's consider a specific case:
\[ m = n = 1. \]

\[ \mathbf{A} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} \]

**Case 1:** \( a \neq 0 \)
Then \( x = \frac{b}{a} \) is the unique solution.

**Case 2:** \( a = 0 \)
(a) \( b \neq 0 \) \( \Rightarrow 0 = b \) (not true)

So, no solution.

(b) \( b = 0 \) \( \Rightarrow \) any \( x \in \mathbb{R} \) is a solution
So, infinitely many solutions.

**Question 1:** How many solutions can a linear system have?
One more special case:

\[ m = n = 2 \]

\[ \begin{align*}
\begin{align*}
& a_{11} x_1 + a_{12} x_2 = b_1 \\
& a_{21} x_1 + a_{22} x_2 = b_2
\end{align*}
\end{align*} \tag{1} \tag{2}
\]

\[ q_{22} x_1 - a_{12} x_2 = 0 \tag{3} \]

\[ (a_{22} a_{11} - a_{12} a_{21}) x_1 = a_{22} b_1 - a_{12} b_2 \]

Case 1: \[ a_{22} a_{11} - a_{12} a_{21} \neq 0 \]

\[ \begin{align*}
\begin{align*}
x_1 &= \frac{a_{22} b_1 - a_{12} b_2}{a_{22} a_{11} - a_{12} a_{21}} \\
x_2 &= \frac{-a_{21} b_1 + a_{11} b_2}{a_{22} a_{11} - a_{12} a_{21}}
\end{align*}
\end{align*} \]

Case 2: \[ a_{22} a_{11} - a_{12} a_{21} = 0 \]

(a) If \[ a_{22} b_1 - a_{12} b_2 \neq 0, \]
    then \[ \boxed{\text{No Solution}} \]

(b) If \[ a_{22} b_1 - a_{12} b_2 = 0, \]
    then \[ \boxed{\text{Infinitely many solutions}} \]