(C2) Differentiability at \( x_2, \ldots, x_{n-1} \)

\[ p_j'(x_{j+1}) = p_{j+1}'(x_{j+1}) \]

for \( j=1, \ldots, n-2 \)

(n-2 equations)

(C3) Twice differentiability at \( x_2, \ldots, x_{n-1} \)

\[ p_j''(x_{j+1}) = p_{j+1}''(x_{j+1}) \]

for \( j=1, \ldots, n-2 \)

(n-2 equations)

Remember:

- \# unknowns = 4n-4
- \# eqns = 4n-6

(C4) Impose \( f''(x_1) = f''(x_n) = 0 \).

\[ \Rightarrow \]

\[ p_1''(x_1) = 0 \]

\[ p_{n-1}''(x_n) = 0 \]

2 more eqns

Now we have

4n-4 eqns = \# unknowns!

Let's do an example:

Ex: Set \( n=3 \); unknowns are \( A_j, B_j, C_j, D_j \) \( j=1,2 \)

(n-2 equations = 8 unknowns)
The corresponding eqns:

(C1) $P_1(x_1) = y_1$:

\[ A_1 x_1^3 + B_1 x_1^2 + C_1 x_1 + D_1 = y_1 \]

\[ P_2(x_2) = y_2: \quad A_2 x_2^3 + B_2 x_2^2 + C_2 x_2 + D_2 = y_2 \]

\[ P_2(x_3) = y_3: \quad A_2 x_3^3 + B_2 x_3^2 + C_2 x_3 + D_2 = y_3 \]

\[ 4 \text{ eqns} \]

(C2) $P_1'(x_2) = P_2'(x_2)$

Noting that

\[ P_j'(x) = 3A_j x^2 + 2B_j x + C_j \]

\[ \Rightarrow 3A_1 x_2^2 + 2B_1 x_2 + C_1 = 3A_2 x_2^2 + 2B_2 x_2 + C_2 \]

\[ P_1'(x_2) \quad P_2'(x_2) \]

A better way of writing this eqn:

\[ 3A_1 x_2^2 + 2B_1 x_2 + C_1 - 3A_2 x_2^2 - 2B_2 x_2 - C_2 = 0 \]

\[ 1 \text{ eqn} \]

(C3) $P_1''(x_2) = P_2''(x_2)$

\[ P_j''(x) = 6A_j x + 2B_j \]

\[ 6A_1 x_2 + 2B_1 = 6A_2 x_2 + 2B_2 \]

\[ \Rightarrow 6A_1 x_2 + 2B_1 - 6A_2 x_2 - 2B_2 = 0. \]

\[ 1 \text{ eqn} \]

(C4) $P_1'''(x_1) = 0$

\[ 6A_1 x_1 + 2B_1 = 0 \]

\[ P_2'''(x_3) = 0 \]

\[ 6A_2 x_3 + 2B_2 = 0 \]

\[ \Rightarrow 8 \text{ eqns}, 8 \text{ unknowns!} \]
Let's write this linear system using matrix notation.

\[
\begin{bmatrix}
 x_1^3 & x_1^2 & x_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 x_2^3 & x_2^2 & x_2 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & x_3^2 & x_3 & x_2 & 1 & 0 \\
 0 & 0 & 0 & 0 & x_3^2 & x_3 & x_3 & 1 & 0 \\
 3x_2^2 & 2x_2 & 1 & 0 & x & x & x & 0 & 0 \\
 6x_2 & 2 & 0 & 0 & -6x_2 & -2 & 0 & 0 & 0 \\
 6x_1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 6x_3 & 2 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
 A_1 \\
 B_1 \\
 C_1 \\
 D_1 \\
 A_2 \\
 B_2 \\
 C_2 \\
 D_2 \\
\end{bmatrix}
= \begin{bmatrix}
 y_1 \\
 y_2 \\
 y_3 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
\end{bmatrix}
\]

Need to solve: \( S a = b \)

- we know \( S \) and \( b \); want to find \( a \), which will then give us \( p_1(x) \) and \( p_2(x) \).