I.2 Interpolation

**Problem:** Given a set of data points \((x_1, y_1), \ldots, (x_n, y_n)\) in \(\mathbb{R}^2\).

**Why do we care?**

- Digital representations of audio, images, etc. — from bitstreams to real signals that we can hear, see, etc.
- Machine learning

**How do we interpolate?**

As stated above, the interpolation problem is NOT well-posed: given any data points, \(f\) infinitely many functions that interpolate.

**Need:** Further restrict the set of allowed functions \(f(x)\)!
LAGRANGE INTERPOLATION

given: \((x_1, y_1), \ldots, (x_n, y_n)\)

Want: A polynomial \(P(x)\) of degree \(n-1\) that fits the data.

Let's find \(P(x)\):

\[
P(x) = a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n
\]

\(n\) unknowns

Let's plug in the data:

\(P(x_1) = y_1, P(x_2) = y_2, \ldots, P(x_n) = y_n\)

\[\begin{align*}
a_1 x_1^{n-1} + a_2 x_1^{n-2} + \cdots + a_{n-1} x_1 + a_n &= y_1 \\
a_1 x_2^{n-1} + a_2 x_2^{n-2} + \cdots + a_{n-1} x_2 + a_n &= y_2 \\
\vdots & \vdots \\
a_1 x_n^{n-1} + a_2 x_n^{n-2} + \cdots + a_{n-1} x_n + a_n &= y_n
\end{align*}\]

A linear system with \(n\) equations \(n\) unknowns: \(a_1, \ldots, a_n\)

Note: \(x_1, x_2, \ldots, x_n\) are known numbers.

Using matrix notation:

\[
\begin{bmatrix}
x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\
x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{n-1} \\
a_n
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
\]

\((\star)\)

\(n \times n\) Vandermonde matrix (known) generated by \(x_1, x_2, \ldots, x_n\)

\(n\) \(a_i\) (unknown)
Ex: The Vandermonde matrix generated by \(2, 3, 5\) is:

\[
\begin{bmatrix}
2^2 & 2^1 & 2^0 \\
3^2 & 3^1 & 3^0 \\
5^2 & 5^1 & 5^0
\end{bmatrix}
= \begin{bmatrix} 4 & 2 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{bmatrix}
\]

Remarks

1. If \([a_1, a_2, \ldots, a_n]^T\) solves the linear system above, then \(p(x) = a_1 x^{n-1} + \cdots + a_{n-1} x + a_n\) fits the data.

2. For a given set of data, is the system (\#) invertible? Well-conditioned?

Answer re: invertibility:

**Theorem:**

\[
\begin{vmatrix}
x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\
x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1
\end{vmatrix}
\]

\[
= (-1)^{n(n-1)/2} \prod_{i \geq j} (x_i - x_j)
\]

\(\Sigma x:\)  If \(n = 3\); \(x_1, x_2, x_3\)

\[
\begin{vmatrix}
x_1^2 & x_1 & 1 \\
x_2^2 & x_2 & 1 \\
x_3^2 & x_3 & 1
\end{vmatrix}
\]
Conclusion: If \( x_1, x_2, \ldots, x_n \) are all distinct, then the corresponding Vandermonde matrix is invertible.

On the other hand, Vandermonde matrices are often not well conditioned – see the MATLAB experiments next class.

**Cubic Spline Interpolation**

(§ 2.3 & § 2.6)