Most commonly used norms in $\mathbb{R}^n$

(1) $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$\| x \|_2 = \left( x_1^2 + x_2^2 + \ldots + x_n^2 \right)^{1/2} = \left( \sum_{j=1}^{n} |x_j|^2 \right)^{1/2}$

This is the "Euclidean norm" or the "$l_2$-norm" of $x^T$.

(2) $\| x \|_1 = \sum |x_j|$

This is the "$l_1$-norm" or the "1-norm".

(3) $\| x \|_p = \left( \sum_{j=1}^{n} |x_j|^p \right)^{1/p}$

This is the "$l_p$-norm" or the "$p$-norm"

wherever $1 \leq p < \infty$

(4) $\| x \|_\infty = \max \left\{ |x_1|, |x_2|, \ldots, |x_n| \right\}$

"$\infty$-norm" or "$l_\infty$-norm"

Fact: $\| x \|_\infty = \lim_{p \to \infty} \| x \|_p$
\[ (1) \ x = [-1, 2, -3]^T \]
\[
\| x \|_2 = \sqrt{(-1)^2 + 2^2 + (-3)^2} = \sqrt{14}
\]

\[ (1) \ x \|_1 = | -1 | + | 2 | + | -3 | = 1 + 2 + 3 = 6 \]

\[ (1) \ x \|_\infty = \max \{| -1 |, | 2 |, | -3 | \} = \max \{1, 2, 3\} = 3 \]

**What makes a norm a norm?**

\( \| x \| \) is a norm if:

1. \( \| x \| \geq 0 \) with \( \| x \| = 0 \iff x = 0 \)
2. For every vector \( v \) and scalar \( s \), \( \| sv \| = | s | \cdot \| v \| \)

(3) For all vectors \( u, v \), we have \( \| u + v \| \leq \| u \| + \| v \| \)

"triangle inequality"

**Exercise (1)** Define \( \| x \| = x_1 t x_2 t \ldots t x_n \)

Is this a norm on \( \mathbb{R}^n \)?

**No!**

- \( \| [-1, -1, -1] \| = -3 < 0 \) !
  Property (1) is violated!
- \( \| [-1, 1, 0] \| = 0 \) !
  Again property (1) is violated.

Exercise: Show that we can violate properties (2) and (3) as well.
Ex. 2: \[ \|x\|_{0.5} = \left( \sum_i |x_i|^{0.5} \right)^{1/0.5} \] (\( l_{0.5} \) - "norm")

This satisfies Property (1).

How about property (2)?

\[ \|sx\|_{0.5} = \left( \sum_{j=1}^{n} |s| |x_j|^{0.5} \right)^2 \]

\[ = (|s| \sum_{j=1}^{n} |x_j|^{0.5})^2 \]

\[ = |s| \sum_{j=1}^{n} |x_j|^{0.5} \]

\[ = |s| \cdot \|x\|_{0.5} \]

Property (2) is also satisfied.

However, property (3) is **not** satisfied — exercise — find counterexamples.

However,

\[ \|x + y\|_{0.5} \leq C_{P,0.5} (\|x\|_{0.5} + \|y\|_{0.5}) \]

In this case, \( \|x\|_{0.5} \) is called a quasi-norm.

I. 1. 6 Norms of matrices

There are many ways to define (useful) norms of matrices. Some important ones:

1. Hilbert–Schmidt (Frobenius)

   Let \( A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \)

   \[ \|A\|_{HS} = \|A\|_F = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \]

   HS norm is easy to compute & useful!
(2) The matrix norm (or the "operator norm")

"Expansion ratio": \( \frac{\|Ax\|_2}{\|x\|_2}, x \neq 0 \)

Then, define

\[ \|A\|_\text{op} := \|A\| = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \]