Remark: For an $n \times n$ system (n eqns, n unknowns), we have
(i) the system $Ax = b$ has a unique solution iff $\det(A) \neq 0$
(ii) if $\det(A) = 0$, then the system has either no solution or inf. many solutions.

Note: We can define the determinant of any $n \times n$ square matrix - see your notes from your earlier linear algebra class or our typed notes.

How do we find the solutions?

Gaussian elimination

Problem: Given

$$Ax = b,$$

find all $x$ that solve this system.

To do Gaussian elimination:

- Build the augmented matrix

$$[A \mid b]$$

- Using elementary row operations, bring this augmented matrix into (reduced) row-echelon form
Ex: Solve for \( x \):

\[
\begin{bmatrix}
3 & 2 & 3 \\
1 & -1 & -4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

So, observe: \( x_3 \) is free.

We have 3 variables, 2 equations.

Choose one variable to be "free": \( x_3 = t \) (we choose \( x_3 \) to be free because the column #3 is not a pivot column).

Then:

\[
x_1 = t + \frac{2}{5}
\]

\[
x_2 = -3t - \frac{3}{5}
\]

So, the solution set is:

\[
\left\{
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
\frac{2}{5} \\
-\frac{3}{5} \\
0
\end{bmatrix} + t \begin{bmatrix}
\frac{1}{3} \\
-1 \\
0
\end{bmatrix} : t \in \mathbb{R}
\right\}
\]

Alternative way of writing:

\[
X = \begin{bmatrix}
\frac{2}{5} \\
-\frac{3}{5} \\
0
\end{bmatrix} + t \begin{bmatrix}
\frac{1}{3} \\
-1 \\
0
\end{bmatrix}
\]

in 3d
Special case: Suppose $A$ is a square matrix, say $n \times n$. Also, suppose $\det(A) \neq 0$. Then for any $y \in \mathbb{R}^n$, there exists a unique $x$ such that

$$A \cdot x = y$$

We then define $A^{-1}$ as the mapping that maps $y$ to $x$. That is

$$A^{-1} \cdot y = x$$

Fact: $A^{-1}$ is also an $n \times n$ matrix!

Define:

$$I_n = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}$$

$n$-d identity matrix

(takes on the role of the number 1 in multiplication).

- $A \cdot I_n = A$ for any $m \times n$ $A$
- $I_n \cdot B = B$ for any $n \times m$ $B$
- $I_n \cdot x = x$ for $x \in \mathbb{R}^n$. 
Let's combine: For $A$ an $n \times n$;
$\det(A) \neq 0$, we have
(i) $A^{-1}$ exists. We say that $A$ is invertible.
(ii) $A^{-1}$ is $n \times n$
(iii) $AA^{-1} = A^{-1}A = I_n$

In this case, the solution of $Ax = b$ is given by
$x = A^{-1}Ax = A^{-1}b$

That is, we have an analytic solution (or formula)!

How do we compute $A^{-1}$?

- Build $[A : I_n]$
  $\rightarrow \ldots [I_n : A^{-1}]$

Norms of vectors & matrices

Norms of vectors (1.15)

Given $x \in \mathbb{R}^n$, find a way of defining the magnitude or size of $x$.
For $n=1$: $x \in \mathbb{R}$, $|x|_1$ does this job!
Most commonly used norms in $\mathbb{R}^n$

(1) $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

$$= \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}$$

This is the "Euclidean norm" or the "$l_2$-norm" of $\vec{x}$.

(2) $\|\vec{x}\|_1 = \sum |x_j|$

This is the "$l_1$-norm" or the "$l_1$-norm".