University of British Columbia  
Math 307, Final

Name:

Student Number:

Signature:

Instructor:

Instructions:

1. No notes, books or calculators are allowed. A MATLAB/Octave formula sheet is provided.

2. Read the questions carefully and make sure you provide all the information that is asked for in the question.

3. Show all your work. Answers without any explanation or without the correct accompanying work could receive no credit, even if they are correct.

4. Answer the questions in the space provided. Continue on the back of the page if necessary.

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1. Suppose you are given a set of $N$ data points $(x_n, y_n)$, with $x_n$ increasing, and you wish to interpolate these points with a spline function $f$, where $f(x)$ is given by a cubic polynomial $p_n(x)$ on each interval $(x_n, x_{n+1})$, for $n = 1, \ldots, N - 1$:

$$p_n(x) = a_n(x - x_n)^3 + b_n(x - x_n)^2 + c_n(x - x_n) + d_n.$$ 

(a) Write down the equations required for $f(x)$ to be continuous and to pass through the data points. How many equations does this provide?

Solution:
$p_n(x_n) = y_n$ and $p_n(x_{n+1}) = y_{n+1}$ for $n = 1, \ldots, N - 1$. $2(N - 1)$ equations.

(b) Write down the equations required for $f(x)$ to have continuous first and second derivatives. How many equations does this provide?

Solution:
$p'_n(x_n) = p'_{n-1}(x_n)$ and $p''_n(x_n) = p''_{n-1}(x_n)$ for $n = 2, \ldots, N - 1$. $2(N - 2)$ equations.
(c) Now suppose \( y_1 = y_N \) and you wish the spline to be periodic, which means adding the two conditions \( f'(x_1) = f'(x_N) \) and \( f''(x_1) = f''(x_N) \). Write down the two equations required for these conditions.

**Solution:**

\[
p_1'(x_1) = p'_{N-1}(x_N) \quad \text{and} \quad p_1''(x_1) = p''_{N-1}(x_N).
\]

(d) Write down the matrix equation to be solved for the coefficients of the polynomials in the case \( N = 3 \), with \( x_1 = 1, x_2 = 2, x_3 = 3 \).

**Solution:**

\[
d_1 = y_1 \quad \text{and} \quad d_2 = y_2 \quad \text{follow straight from (a). The other coefficients satisfy} \]

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
3 & 2 & 1 & 0 & 0 & -1 \\
6 & 2 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & -1 & 3 & 2 & 1 \\
0 & -2 & 0 & 6 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
a_1 \\
b_1 \\
c_1 \\
a_2 \\
b_2 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
y_2 - y_1 \\
0 \\
0 \\
y_3 - y_1 \\
0 \\
0
\end{bmatrix}
\]
2. Suppose that you want to solve the boundary value problem
\[
f''(x) - x^2 f(x) = 1 \quad 0 \leq x \leq 1, \quad f(0) = 1, \quad f(1) = 1.
\]
Let \( F = [f_0, f_1, \ldots, f_N]^T \) be the finite difference approximation to the solution \( f(x) \) at uniformly spaced points \( x_0, x_1, \ldots, x_N \).

(a) Write down finite difference approximation to \( f''(x) \) at an interior point \( x_i \).

\[
\text{Solution:} \quad f''(x_i) = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2}
\]
where \( \Delta x = x_1 - x_0 \) is the uniform grid spacing.

(b) The finite difference approximation to the equations and boundary conditions can be written in the matrix form \( AF = b \). Find the matrix \( A \) and the vector \( b \) for the case \( N = 4 \).

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & -2 - \Delta x^2 x_1^2 & 1 & 0 \\
0 & 1 & -2 - \Delta x^2 x_2^2 & 1 \\
0 & 0 & 1 & -2 - \Delta x^2 x_3^2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
f_0 \\
f_1 \\
f_2 \\
f_3 \\
f_4
\end{bmatrix} = \begin{bmatrix}
1 \\
\Delta x^2 \\
\Delta x^2 \\
\Delta x^2 \\
1
\end{bmatrix}
\]
(c) Suppose the boundary condition at \( x = 0 \) is changed to

\[
f(0) = 1 - f'(0).
\]

How do \( A \) and \( b \) change in this case?

**Solution:**

The first row of \( A \) becomes

\[
\begin{bmatrix}
-1 + \Delta x & 1 & 0 & 0
\end{bmatrix}
\]

and the first entry of \( b \) becomes \( \Delta x \).
3. Suppose that $A$ is a $3 \times 4$ matrix and

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For each of the following, either find what is asked for, or indicate that there is insufficient information to determine it: (i) the rank of $A$, (ii) a basis for $N(A)$, (iii) a basis for $R(A)$, (iv) a basis for $N(A^T)$, (v) a basis for $R(A^T)$.

**Solution:**

(i) The rank of $A$ is 2.

(ii) A basis for $N(A)$ is \{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}\}

(iii) There is insufficient information to determine a basis for $R(A)$.

(iv) There is insufficient information to determine a basis for $N(A^T)$.

(v) A basis for $N(A)$ is \{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}\}
4. Consider a resistor network arranged in the shape of an octahedron as in this diagram:

The nodes have been labeled with the large numbers and 3 of the edges (resistors) have been given orientations and labeled with small numbers. Assume all resistances are equal to 1. Let $D$ be the incidence matrix (for some choice of labeling and arrows for the remaining edges) and let $L$ be the Laplacian.

(a) Write down a non-zero vector in $N(D)$

Solution:
A non-zero vector in $N(D)$ is $[1, 1, 1, 1, 1]^T$.

(b) Write down a non-zero vector in $N(D^T)$

Solution:
A non-zero vector in $N(D^T)$ is the loop vector corresponding to the loop around edges 1, 2 and 3. Taking account of the orientation we get $[1, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0]^T$.

(c) Is it true that $N(D) = N(L)$? Give a reason.

Solution:
Yes. If $Dx = 0$ then $Lx = D^TDx = D^T0 = 0$. Conversely, if $Lx = 0$ then $\|Dx\|^2 = (Dx) \cdot (D^TDx) = x \cdot (Lx) = x \cdot 0 = 0$ so $Dx = 0$.

(d) Write down the Laplacian $L$.

Solution:
$L = \begin{bmatrix}
4 & -1 & -1 & -1 & -1 & 0 \\
-1 & 4 & -1 & 0 & -1 & -1 \\
-1 & -1 & 4 & -1 & 0 & -1 \\
-1 & 0 & -1 & 4 & -1 & -1 \\
-1 & -1 & 0 & -1 & 4 & -1 \\
0 & -1 & -1 & -1 & -1 & 4
\end{bmatrix}$
(e) It seems reasonable to conjecture that the effective resistance between nodes 1 and 6 would remain unchanged if we removed the resistors between 2 and 3, 3 and 4, 4 and 5, 5 and 2. Explain how you could use MATLAB/Octave to test this conjecture. Assume that $L$ has been defined in MATLAB/Octave.

**Solution:**

First compute the effective resistance $R$ between nodes 1 and 6 for the original graph.

```matlab
LL=L([1,6,2:5],[1,6,2:5])
A=LL(1:2,1:2)
B=LL(3:6,1:2)
C=LL(3:6,3:6)
DN = (A-B'*C^(-1)*B)
R=1/DN(1,1)
```

To compute the effective resistance $R_1$ with the removed resistors we could notice that the resulting circuit is equivalent to four resistors with resistance 2 connected in parallel. Thus $1/R_1 = 1/2 + 1/2 + 1/2 + 1/2$. This gives $R_1 = 1/2$ which we can compare with $R$.

Alternatively, modify $L$ to reflect the changes:

```matlab
L(2,2)=L(3,3)=L(4,4)=L(5,5)=2
L(2,3)=L(3,2)=L(3,4)=L(4,3)=L(4,5)=L(5,4)=L(5,2)=L(2,5)=0
```

Then repeat the steps above to compute $R_1$ which can then be compared with $R$. 
5. You are given a set of 100 data points \((x_n, y_n)\) with \(x_n\) increasing.

(a) Suppose you wish to use Lagrange interpolation to interpolate the data points. What degree of polynomial would you use and why?

**Solution:**

99 degree polynomial, since this will have 100 unknown coefficients to choose from the 100 data points.

(b) Write down the matrix equation that is satisfied by the coefficients of this polynomial. Is the numerical solution of this equation likely to be accurate, and why?

**Solution:**

If the polynomial is \(c_0x^{99} + c_1x^{98} + \ldots + c_{99}\) then

\[
\begin{bmatrix}
x_1^{99} & x_1^{98} & \ldots & x_1 & 1
x_2^{99} & x_2^{98} & \ldots & x_2 & 1
ds & ds & \ldots & ds & 1
ds & ds & \ldots & ds & 1
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_{99}
\end{bmatrix}
= 
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_{100}
\end{bmatrix}
\]

Numerical solution is not likely to be accurate because the Vandermonde matrix \(V\) has a large condition number.
(c) Suppose instead that you choose to make a least squares fit of the data points to a quadratic polynomial

\[ f(x) = c_0 x^2 + c_1 x + c_2. \]

Write down the Matlab/Octave commands you would use to calculate the coefficients, plot the original data points and plot the quadratic fit (you may assume that vectors \( X = [x_1, x_2, \ldots, x_N]^T \) and \( Y = [y_1, y_2, \ldots, y_N]^T \) have already been defined).

**Solution:**

The equation to solve in this case is \( A^T A c = A^T Y \), where \( c = [c_0, c_1, c_2]^T \) and \( A \) is the last three columns of the Vandermonde matrix in part (b).

\[
> A = \begin{bmatrix} X.\text{^2} & X.\text{^1} & X.\text{^0} \end{bmatrix};
> c = (A'*A) \backslash (A'*Y);
> XL = linspace(X(1),X(100),1000); \quad % \text{prepare to plot the fit at 1000 points}
> YL = polyval(c,XL);

> plot(X,Y,'o'); hold on;
> plot(XL,YL,'r');
6. (a) A continuous measurement of the temperature in Vancouver, \( y(t) \), for \( 0 \leq t \leq T \), is decomposed as a Fourier series of the form \( y(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i\omega_n t} \), where \( \omega_n = n/T \).

What important property do the set of functions \( \{e^{2\pi i\omega_n t}\} \) have? What is the expression for the Fourier coefficients \( c_n \) in terms of \( y(t) \)?

**Solution:**

These functions are orthogonal so \( \langle e^{2\pi i\omega_n t}, e^{2\pi i\omega_m t} \rangle = 0 \) if \( m \neq n \).

The expression for \( c_n \) is

\[
 c_n = \frac{1}{T} \int_0^T e^{-2\pi i\omega_n t} y(t) \, dt
\]

(b) Explain the physical significance of the values \(|c_n|\) and how these relate to \( \omega_n \). If \( t \) is measured in days, and the time period \( T \) encompasses several years, which values of \( c_n \) might you expect to have the largest absolute value? (Think about what time scales you expect temperature to fluctuate over).

**Solution:**

\(|c_n|\) represents the amplitude of frequency \( \omega_n \) in the signal \( y(t) \) - it is the amount of the wave with that frequency when the signal is decomposed as a superposition of waves with different frequencies.

Expect dominant frequencies in \( y(t) \) to be 1 day\(^{-1}\) and 1 year\(^{-1}\); i.e. \( \omega_n \approx 1 \) and \( \omega_n \approx 1/365 \) when \( t \) is measured in days. So the largest absolute values of \( c_n \) will be for \( n \approx T \) and \( n \approx T/365 \).
(c) Now suppose a vector $y$ contains measurements of the daily average temperature for 10 years. Explain how to use the Discrete Fourier Transform to produce a frequency-amplitude plot for this data. You should write the Matlab/Octave commands you would use, and include the range of frequencies 0 to 0.1 day$^{-1}$ on your plot.

**Solution:**
Find the discrete Fourier transform of the signal and approximate the Fourier coefficients $c_n$ by the corresponding entries in the discrete Fourier transform (divided by $N$, the length of the signal). Plot these amplitudes against the frequencies $\omega_n = n/N$ and then restrict the range of the axes.

> c = fft(y); % discrete Fourier transform
> N = length(c); % length of signal
> Fs = 1; % sampling frequency
> omega = (0:N-1)/N*Fs;
> plot(omega,abs(c)/N);
> xlim([0 0.1]); % change axes to plot only frequencies 0 to 0.1 day$^{-1}$

(d) Sketch the plot that you expect to be produced by your answer to part (c).

**Solution:**
The picture might look something like this, with a peak near $\omega = 1/365$:
7. Consider the MATLAB/Octave computation

1> A=[0 2 1;1 0 0 ;0 1 0]
A =
0 2 1
1 0 0
0 1 0

2> [S D]=eig(A)
S =
0.80902 -0.57735 0.30902
0.50000 0.57735 -0.50000
0.30902 -0.57735 0.80902

D = Diagonal Matrix

1.61803 0 0
0 -1.00000 0
0 0 -0.61803

(a) Write down a recursion relation for a sequence \(x_0, x_1, x_2, \ldots\) that you can analyse using this calculation. How many initial values do you have to specify to define the sequence?

**Solution:**
Specify three initial values \(x_0, x_1\) and \(x_2\). Then for \(n \geq 2\),

\[x_{n+1} = 2x_{n-1} + x_{n-2}\]

(b) If you picked initial values at random, what would you expect the large \(n\) behaviour of the resulting sequence to be? Give a reason.

**Solution:** Since a random initial vector \([x_0, x_1, x_2]^T\) would have a component in the direction of the eigenvector whose eigenvalue is \(\lambda_1 = 1.61803\)… the expected leading behaviour would be \(x_n \sim c\lambda_1^n\)

(c) Write down initial values which result in a non-zero periodic sequence where the \(x_n\) repeatedly cycle through the same values.

**Solution:** The second eigenvalue is \(-1\) so if we pick initial values proportional to the corresponding eigenvector the values will oscillate proportionally to \((-1)^n\). For example, picking \(x_0 = 1, x_1 = -1, x_2 = 1\) we find \(x_3 = -1, x_4 = 1, \ldots, x_n = (-1)^n\).
8. Consider the stochastic matrix

\[ P = \begin{bmatrix}
0 & 1/4 & 1/3 \\
1 & 1/2 & 1/3 \\
0 & 1/4 & 1/3
\end{bmatrix} \]

(a) The vector \([3, 8, 3]^T\) is an eigenvector of \(P\). What is the eigenvalue?

**Solution:**

\[
P \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 + 1 \\ 3 + 3 + 1 \\ 2 + 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix}
\]

so the eigenvalue is 1.

(b) By considering \(P^2\), what can you say about the other eigenvalues of \(P\)? Give a reason.

**Solution:** Since \(P^2\) has strictly positive entries, the other eigenvalues satisfy \(|\lambda_2| < 1, |\lambda_3| < 1\).

(c) What is \(\lim_{n \to \infty} P^n[1, 0, 0]^T\)? Give a reason.

**Solution:** The limit is the multiple of \([3, 8, 3]^T\) whose entries sum to 1. So

\[
\lim_{n \to \infty} P^n[1, 0, 0]^T = \frac{1}{14}[3, 8, 3]^T.
\]