Cylindrical coordinates & triple integrals (secondary text #2: 14.4)

The point \( P \) can be represented by \((r, \theta)\) and \( z \)

**Def:** \((r, \theta, z)\) are the cylindrical coordinates of the point \( P = (x, y, z) \).

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  z &= z
\end{align*}
\]

\[
\begin{align*}
  r^2 &= x^2 + y^2 \\
  z &= z
\end{align*}
\]

\( \theta = \tan^{-1}(y/x) \)

Note: \( r \) is the distance of the point \( P \) from the \( z \)-axis.

Examples of regions easily described using cylindrical coordinates.

1. \( r = c \); \( c > 0 \), constant
   - Cylinder with radius \( c \).

2. \( \theta = \text{const} \)
(3) $z = \text{const}$

(4) \[ z = \sqrt{x^2 + y^2} \]

Since we have \[ r^2 = x^2 + y^2, \]

this cone can be described as:

\[ z = \sqrt{r^2} = r \]

(5)

Ex: Describe the following solids using cylindrical coord.

(i) \[ z = 2 \quad \text{and} \quad x^2 + y^2 = 4 \]

In cylindrical coord:

\[ 0 \leq z \leq 2 \]
\[ 0 \leq r \leq 2 \]
\[ 0 \leq \theta \leq 2\pi \]
(ii) $2 = \frac{x}{2} \quad x = 0$
$x^2 + y^2 = 4$
$z = 0$

$0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq \phi \leq \frac{\pi}{2}$

(Note: both (i) and (ii) are cylindrical "rectangles")

(iii) $z = 2$
$x^2 + y^2 = 2z$

(iv) $z = \sqrt{1-r^2}$
$z = -\sqrt{1-r^2}$

Sphere:
$-\sqrt{1-r^2} \leq z \leq \sqrt{1-r^2}$
$0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$
Want to write triple integrals in cylindrical coord.: 

$$I = \iiint_R f(r, \theta, z) \, dV$$

Three tasks:

1. Describe $R$ using cylindrical coord. (what we just did!)
2. Write $f(r, \theta, z)$ as a function of $r, \theta, z \rightarrow$ straight forward.
3. Evaluate $dV$ in terms of cylindrical coord.

$\uparrow$ let's do this!

$$dV = r \, d\theta \, dr \, dz$$

Most of the time: we will stick to the order

$$dV = r \, dz \, dr \, d\theta$$

Then: if $R$ is described as:

$$f_1(r, \theta) \leq z \leq f_2(r, \theta)$$

$$g_1(\theta) \leq r \leq g_2(\theta)$$

$$a \leq \theta \leq b$$

$$\iiint_R f(x, y, z) \, dV = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r, \theta)}^{f_2(r, \theta)} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta$$
Examples: Look at our previous example:

\[ z = y \]
\[ x^2 + y^2 = 1. \]

Volume:

\[
\iiint_R dV = \int_0^{2\pi} \int_{-1}^{r \sin \theta} \int_0^1 r \, dz \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^1 r^2 \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left[ \frac{1}{2} r^2 \right]_0^1 \, d\theta
\]

\[
= \frac{1}{2} \theta \bigg|_0^{2\pi} = \pi \text{ \text{ \text{ \text{ \text{\text{II}}}}}}
\]