Want: to convert \( \iint f \, dA \) to an (iterated) integral in polar coordinates.

For this purpose:

(i) Need to express \( f(x, y) \) as a function of \( r \) and \( \theta \):
- Just subst. \( x = r \cos \theta \); \( y = r \sin \theta \)
- Ex. \( f(x, y) = x + y^2 = r \cos \theta + r^2 \sin^2 \theta \)

(ii) Need to describe \( R \) using polar coordinates.
- \( \theta \) is \( 0 \) to \( \frac{5\pi}{4} \)
- \( \Delta_r \) is \( 0 \) to \( 2 \)
- \( \Delta_\theta \) is \( \frac{\pi}{6} \) to \( \frac{2\pi}{3} \)

Ex. \( R \) is a quarter circle with center at \( (0, 0) \) and radius 2.
Ex: 
\[(x-1)^2 + y^2 = 1\]

First, describe the circle using polar coordinates:
\[(r \cos \theta - 1)^2 + (r \sin \theta)^2 = 1\]
\[r^2 \cos^2 \theta + r^2 \sin^2 \theta - 2r \cos \theta + 1 = 1\]
\[r^2 - 2r \cos \theta = 0\]
\[r = 2 \cos \theta\] (\[\Rightarrow\] not a polar rectangle!)

Disk: \[0 \leq r \leq 2 \cos \theta\]
\[-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\]

Exercise: Describe
\[x^2 + (y-1)^2 = 1\]

(iii) Need to express \(dA\) using polar coord.

Recall: with Cartesian coord:
\[dA = dx dy\]
\[dA = \frac{dA}{dy} dy\]
With polar coord.: partition into infinitesimal (really small) polar rectangles!

\[ \text{Conclusion: } dA = r \, dr \, d\theta \]

**Examples**

1. Find the volume of the solid bounded by the plane \( z = 0 \) and the paraboloid \( z = 1 - x^2 - y^2 \)

**Sol.**

\[ V = \iiint_R 1 - x^2 - y^2 \, dA \]

\[ R: \text{ Set } 0 = 1 - x^2 - y^2 \]  
\[ \Rightarrow x^2 + y^2 = 1 \]  
\[ \text{boundary of } R \]

\[ \text{intersection of } z = 0 \]

\[ \text{with the paraboloid} \]
\[ R = \{ (x, y) : x^2 + y^2 \leq 1 \} . \]

Also:
\[ f(x, y) = 1 - x^2 - y^2 = 1 - (x^2 + y^2) = 1 - r^2 \]

Then:
\[ V = \int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr \, d\theta \]

Ex: (Final, 2014) The disk \( x^2 + y^2 \leq 2x \) is divided into two regions by the line \( y = x \). Let \( R \) be the lower one.

(a) Sketch \( R \) in the \( xy \)-plane.

(b) Find the volume under \( z = \sqrt{x^2 + y^2} \) over \( R \).

Sol: (a) \( x^2 + y^2 = 2x \) at boundary of the disk
\[ x^2 - 2x + y^2 = 0 \]
\[ \Rightarrow (x-1)^2 - 1 + y^2 = 0 \]
\[ \Rightarrow (x-1)^2 + y^2 = 1 . \]
We next will describe $R$ using polar coordinates:

From before: the boundary circle is: $r = 2 \cos \theta$

Then: $0 \leq r \leq 2 \cos \theta$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{4}$

Then:

$$z^{2} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \int_{0}^{2 \cos \theta} r \, dr \, d\theta$$

Then:

$$z = \sqrt{x^{2} + y^{2}} = \sqrt{r^{2}} = r$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \left( \int_{0}^{2 \cos \theta} r^{2} \, dr \right) \, d\theta = \ldots$$