Ex: Write the volume under $f(x,y) = e^{x^2}$ over the region $R$.

Alternative:

$$V = \int_{0}^{x} \int_{y}^{-x} e^{x^2} \, dy \, dx$$

Soli:

$$V = \int_{0}^{1} \int_{y}^{-y} e^{x^2} \, dx \, dy = \int_{0}^{1} \int_{y}^{-y} e^{x^2} \, dy \, dx$$

First:

$$\int_{0}^{1} \int_{y}^{-y} e^{x^2} \, dx \, dy$$

over $R_1$.

$$\int_{0}^{1} \int_{y}^{-y} e^{x^2} \, dy \, dx$$

over $R_2$.

* typo in class
Double Integrals

Want: Volume under \( z = f(x,y) \) over some region \( R \) in the xy-plane.

In 1-variable case (Math 101/103/05)

\[
\text{height} = f(x_i)
\]

\[
\int_a^b f(x) \, dx \rightarrow \text{Area as } \Delta x \to 0.
\]

Riemann sums

Let's extend this idea to \( f(x,y) \):

Approach:

1. partition \( R \) into small rectangles with sides \( \Delta x \) and \( \Delta y \); label these with \( i = 1, 2, \ldots, n \)

2. In the \( i \)th rectangle, pick some point \( (x_i, y_i) \); consider the box above this rectangle with height \( f(x_i, y_i) \) - this box has volume \( \Delta V = f(x_i, y_i) \cdot \Delta x \cdot \Delta y \)
To obtain the full volume
\[ (3) \quad V = \sum_{i=1}^{n} f(x_i, y_i) \Delta x \Delta y \quad \text{Riemann sum} \]
approximately the volume we are trying to calculate.

Now, take \( n \to \infty \) and \( \Delta x, \Delta y \to 0 \).
If this limit exists, we write
\[ \int_{R}^{\text{"double integral of } f \text{ over the region } R \"}} \]
and gives us the signed volume under \( f \) over \( R \). In this case, we denote this limit by \( \iint_{R} f \, dA \).

**Fubini’s Theorem:** let \( f(x,y) \) be continuous over a closed & bounded region \( R \). Then
\[ (1) \quad \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i) \Delta x \Delta y \quad \text{exists} \]
and gives us the signed volume under \( f \) over \( R \).

(2) If \( R = \{(x, y) : a \leq x \leq b ; \ g(x) \leq y \leq h(x) \} \)
then
\[ \int_{R} f \, dA = \int_{a}^{b} \left( \int_{g(x)}^{h(x)} f(x,y) \, dy \right) \, dx \]

(3) If \( R = \{(x, y) : c \leq y \leq d ; \ h_{1}(y) \leq x \leq h_{2}(y) \} \)
then
\[ \int_{R} f \, dA = \int_{c}^{d} \left( \int_{h_{1}(y)}^{h_{2}(y)} f(x,y) \, dx \right) \, dy \]

Note: In this defn, there is no order of integration! The number we obtain is the "signed" volume of \( \approx f(x,y) \) over \( R \).
Example: Evaluate \( \iiint_R (y - y) \, dA \) where \( R \) is the region bounded by parabolas \( y^2 = 4xy \) and \( x^2 = 4y \).

**Sol:**

Intersection points:

\[
y = \frac{x^2}{4} \quad \text{into} \quad y^2 = 4x\]

gives:

\[
\frac{x^4}{16} = 4x \quad \Rightarrow \quad x(x^3-64) = 0
\]

\( x_1 = 0 \) and \( x_2 = 4 \).

Then:

\[
V = \iiint_R (y - y) \, dA
\]

\[
= \int_0^4 \left( \int_0^{x/4} y \, dy \right) \, dx = \frac{176}{15}
\]

Ex: Calculate \( I = \int_0^3 \int_y^3 e^{-x^2} \, dx \, dy \).

**Sol:** Noting that there is no elementary antiderivative of \( e^{-x^2} \), we must change the order of integration.

\[
:\text{answer} = \frac{1}{2} - e^{-9/2}
\]