Lagrange Multipliers (14.8 from secondary text #1)

Problem: Maximize or minimize a function \( f(x, y) \) over all \( (x, y) \) in \( S \) where \( S \) is a curve in \( \mathbb{R}^2 \).

(Later we'll generalize to functions of more variables!)

Example:

\[
\begin{align*}
\text{Maximize/minimize} & \quad f(x, y) = x^2 + y^2 - 2x + 2y + 5 \\
\text{subject to} & \quad x^2 + y^2 = 4 \quad \text{A constraint}
\end{align*}
\]

\[ \iff \] \( \begin{align*}
\text{max./min.} & \quad f(x, y) \\
\text{s.t.} & \quad (x, y) \in S = \{(x, y): x^2 + y^2 = 4\}
\end{align*} \]

\[ \begin{align*}
\text{define: } & \quad g(x, y) = x^2 + y^2 - 4 \\
\text{max./min.} & \quad f(x, y) \\
\text{s.t.} & \quad g(x, y) = 0
\end{align*} \]

Observations & Remarks

1. At the abs. max. and abs. min. of \( f \) on the curve \( S \), the level curves of \( f \) are tangent to the constraint curve \( S \).

2. For any \( (a, b) \),
   \[ \nabla f(a, b) \perp \text{level curve of } f(x, y) = f(a, b) \]
   \[ \nabla g(a, b) \perp \text{level curve of } g(x, y) = g(a, b) \]
Then if \((a, b)\) is an abs. max or abs. min of \(f(x, y)\) on 
\(S = \{(x, y) : g(x, y) = 0\}\) (a level curve of \(g\) given by \(g(x, y) = 0\)), 
we have: 
\[ \nabla f(a, b) \parallel \nabla g(a, b). \]
\[ \Leftrightarrow \nabla f(a, b) = \lambda \cdot \nabla g(a, b), \ \lambda \in \mathbb{R}. \]

\[ \Leftrightarrow \begin{align*}
    f_x(a, b) &= \lambda g_x(a, b) \\
    f_y(a, b) &= \lambda g_y(a, b)
\end{align*} \]
These observations justify the method of Lagrange multipliers:

**Problem:** max/min \( f(x, y) \) 
subject to \( g(x, y) = C \)

**Solution:** Find \( x, y, \) and \( \lambda \) s.t.
\[
\begin{cases}
    \nabla f(x, y) = \lambda \nabla g(x, y) \\
    g(x, y) = C
\end{cases}
\]

\[
\begin{align*}
    f_x(x, y) &= \lambda g_x(x, y) \\
    f_y(x, y) &= \lambda g_y(x, y)
\end{align*} \] 3 eqns; 
\( g(x, y) = C \) \[ 3 \text{ unknowns!} \] 

**Step 1:** Solve the above system

**Step 2:** Among the solutions obtained in Step 1,
- largest corresponding function value \(
\rightarrow \text{abs. max}\)
- smallest corresponding function value \(
\rightarrow \text{abs. min}\)

on the constraint \( S \).