Gradient & tangent planes (12.7)

(Note: we will not emphasize the material on tangent normal lines in the beginning of 12.7 - please read.)

Facts:

1. Given \( f(x,y) \), \( \nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \)

   is perpendicular to the level curve of \( f \) at \( (a,b) \), given by

\[ f(x,y) = f(a,b). \]

Ex: \( f(x,y) = x^2 + y^2 \)
\[ (a,b) = (2,3) \]
\[ x^2 + y^2 = 13 \quad \Rightarrow \quad f(x,y) = 13; \text{ the level curve that passes through} \]
\[ \nabla f(2,3) = \langle f_x(2,3), f_y(2,3) \rangle \]
\[ = \langle 4, 6 \rangle \]

So, \( \langle 4, 6 \rangle \) is perpendicular to the circle \( x^2 + y^2 = 13 \) at \( (2,3) \)
2. Given \( f(x,y,z) = d \), a level surface of \( f(x,y,z) \), say passing through \((a,b,c)\). Then \( \nabla f(a,b,c) \) is perpendicular to this surface at \((a,b,c)\).

Accordingly, \( \nabla f(a,b,c) \) is the normal vector to the tangent plane to the surface \( f(x,y,z) = d \) at \((a,b,c)\).

Why? Let’s justify Fact ①:

\[
f(x,y) = f(a,b) \quad (*)
\]

Differentiate \((*)\) implicitly w.r.t. \( x \):

\[
f_x(x,y) \frac{dx}{dx} + f_y(x,y) \frac{dy}{dx} = 0.
\]

\[(**)

\[
f_x(a,b) \cdot 1 + f_y(a,b) \frac{dy}{dx} \bigg|_{x=a, y=b} = 0.
\]

Recall:

That is:

\[
\langle 1, \frac{dy}{dx} \rangle \parallel \text{the tangent line}
\]

\[(***) \text{ says: } \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle 1, \frac{dy}{dx} \rangle = 0
\]

\[
\Rightarrow \nabla f(a,b) \perp \text{tangent line}!
\]

② is similar!
Ex: (434 in the book)
(a) Find the equation of the tangent plane to the ellipsoid \( \frac{x^2}{12} + \frac{y^2}{6} + \frac{z^2}{4} = 1 \) at \( P = (1,2,1) \).
(b) Find the equation of the normal line to the ellipsoid at \( P = (1,2,1) \).

Soli:

(a) \( f(x,y,z) = \frac{x^2}{12} + \frac{y^2}{6} + \frac{z^2}{4} \).

- Point on the tangent plane: \( P = (1,2,1) \)
- \( \vec{n} = \nabla f(1,2,1) = ? \)
  - \( f_x(x,y,z) = 2x \cdot \frac{1}{12} = \frac{x}{6} \)
  - \( f_y(x,y,z) = 2y/6 = y/3 \)
  - \( f_z(x,y,z) = 2z/4 = z/2 \)
- \( \nabla f(1,2,1) = \langle \frac{1}{6}, \frac{2}{3}, \frac{1}{2} \rangle = \vec{n} \)

Then, the equation of the plane:

\[
\frac{1}{6} (x - 1) + \frac{2}{3} (y - 2) + \frac{1}{2} (z - 1) = 0
\]

\[\Rightarrow \langle x - 1, 4(y - 2), 3(z - 1) \rangle = 0\]

(b) Point on the line: \( (1,2,1) \)
Direction vector: \( \vec{v} = \langle \frac{1}{6}, \frac{2}{3}, \frac{1}{2} \rangle \)

\( \vec{l}(t) = (1,2,1) + t \cdot \langle \frac{1}{6}, \frac{2}{3}, \frac{1}{2} \rangle \)

Earlier: We considered surfaces \( z = f(x,y) \)

\( \vec{n} = \langle f_x(a,b), f_y(a,b), -1 \rangle \)

Comes from:

\( z = f(x,y) \Rightarrow f(x,y) - z = 0 \)

\( \vec{n} = \nabla F(a,b, f(a,b)) \)

= \( \langle f_x(a,b), f_y(a,b), -1 \rangle \)