**Def:** (1) Given \( f(x,y) \) defined around \((a,b)\), \( \nabla f(a,b) := \langle f_x(a,b), f_y(a,b) \rangle \) is the gradient of \( f(x,y) \) at \((a,b)\).

(2) \( \text{D}u f(a,b) = \nabla f(a,b) \cdot \hat{u} \)

**Facts:**

(1) \( \text{D}u f(a,b) \) is maximum when \( \hat{u} = \hat{u}_0 = \frac{\nabla f(a,b)}{||\nabla f(a,b)||} \) and \( \hat{u} \) is the direction of steepest ascent.

* The maximum rate of change in \( f \) at \((a,b)\) is

\[
\text{D}u \hat{u}_0 f(a,b) = \nabla f(a,b) \cdot \hat{u}_0
\]

\[
= \nabla f(a,b) \cdot \frac{\nabla f(a,b)}{||\nabla f(a,b)||} = \frac{||\nabla f(a,b)||^2}{||\nabla f(a,b)||} = ||\nabla f(a,b)||
\]

(2)* \( \text{D}u f(a,b) \) is minimum when \( u = -\hat{u}_0 = -\frac{\nabla f(a,b)}{||\nabla f(a,b)||} \) and \( u \) is the direction of steepest descent.

* The min. rate of change

\[
\text{D}u_{-\hat{u}_0} f(a,b) = -||\nabla f(a,b)||
\]

**Remark:** The above definitions and facts generalize to functions with more than 2 variables: \( f(x_1, x_2, \ldots) \)

* \( \nabla f(a_1, a_2, \ldots) = \langle f_x(a_1, a_2, \ldots), \ldots \rangle \)

* \( \text{D}u f(a_1, a_2, \ldots) = \nabla f(a_1, a_2, \ldots) \cdot \hat{u} \).

* max. rate of change in the direction of \( \nabla f/||\nabla f|| \)

* min rate of change: \(-\nabla f/||\nabla f||\).
Examples:

1. Suppose that $D_{\left<\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right>} f(0,0) = 0$ (***)
   $D_{\left<-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right>} f(0,0) = 1$ (***)

   (a) $\nabla f(0,0) = ?$

   (b) Direction and amount of max. increase?

   Sol.

   (a): $\nabla f(0,0) = \langle a, b \rangle$. Want: find $a$ and $b$.

   From (**):

   $D_{\left<\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right>} f(0,0) = \nabla f(0,0) \cdot \left<\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right>
   = \langle a, b \rangle \cdot \left<\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right>$
   $= 0$

   $\Rightarrow a \cdot \frac{1}{\sqrt{2}} + b \cdot \frac{1}{\sqrt{2}} = 0 \Rightarrow \boxed{a + b = 0}$ (1)

   From (**):

   $D_{\left<-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right>} f(0,0) = \langle a, b \rangle \cdot \left<-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right>
   = 1$

   $\Rightarrow 1 - a + 2b = \sqrt{5}$ (2)

   Then (1) gives: $a = -b$.

   into (2): $3b = \sqrt{5} \Rightarrow b = \frac{\sqrt{5}}{3}$

   $\Rightarrow a = -\frac{\sqrt{5}}{3}$

   So, $\nabla f(0,0) = \left<-\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{3}\right>$.

   (b) Direction of max increase:

   $\hat{u} = \frac{\nabla f(0,0)}{||\nabla f(0,0)||} = \frac{1}{\frac{\sqrt{2}}{2}} \langle -1, 1 \rangle$

   Max. rate of change $= ||\nabla f|| = \frac{\sqrt{10}}{3}$.
Given: \( P = (2,1,1) \). A function \( T(x,y,z) \)
s.t. \( T(2,1,1) = T(P) = 5 \)
\( T_x(P) = 1 \); \( T_y(P) = 2 \); \( T_z(P) = 3 \)

(a) A bee starts flying at \( P \) (along the unit vector pointing towards \( Q = (3,2,2) \)). What is the rate of change in \( T(x,y,z) \) experienced by the bee?

(b) Use linear approximation at \( P \) to approximate \( T(1.9,1.1,1.2) \).

(c) Let \( S(x,y,z) = x + z \). A bee starts flying at \( P \). Along which unit vector direction should the bee fly so that the rate of change in \( T(x,y,z) \) and in \( S(x,y,z) \) are both 0 in this direction?

(d) In what direction (from \( P \)) does \( T \) increase most rapidly? At what rate of change?

Sol:
(a) We are asked to find
\[
\nabla T(2,1,1) = \nabla T(P) \cdot \vec{u}
\]
where \( \vec{u} = \frac{\vec{PQ}}{||\vec{PQ}||} \)

Find \( \vec{u} \):
\[
\vec{u} = \frac{\langle 3-2, 2-1, 2-1 \rangle}{\sqrt{3}} = \frac{1}{\sqrt{3}} \langle 1,1,1 \rangle
\]

Find \( \nabla T(P) \):
\[
\nabla T(P) = \langle T_x(P), T_y(P), T_z(P) \rangle = \langle 1,2,2 \rangle
\]

Then:
\[
\nabla T(P) = \langle 1,1,1 \rangle \cdot \left( \frac{1}{\sqrt{3}} \langle 1,1,1 \rangle \right) = 0
\]

(b) Trivial - do it as an exercise. (as you did in the midterm)

(c) Find unit vector \( \vec{u} = \langle a, b, c \rangle \) s.t.
\[
D_u T(P) = 0 \quad (1)
\]
\[
D_u S(P) = 0 \quad (2)
\]
(1) \[ \nabla T(P) \cdot \hat{u} = 0 \]
\[ \iff \langle 1, 2, 3 \rangle \cdot \langle a, b, c \rangle = 0 \]
\[ \iff a + 2b + 3c = 0 \]
\[ \iff \frac{a + 2b + 3c}{\sqrt{14}} = 0 \]
\[ (1') \]

(2) \[ \nabla S(x, y, z) = \langle S_x, S_y, S_z \rangle \]
\[ = \langle 1, 0, 1 \rangle \]

(2) \[ \nabla S(P) \cdot \hat{u} = 0 \]
\[ \iff \langle 1, 0, 1 \rangle \cdot \langle a, b, c \rangle = 0 \]
\[ \iff a + c = 0 \]
\[ (2') \]

(1') & (2') has one free variable:
set \( c = s \); \( a = -s \); \( b = -s \)

So, \( \hat{u} = S \langle -1, -1, 1 \rangle \); \( S =? \)

Impose \( \| \hat{u} \| = |S| \sqrt{13} = 1 \)
\[ \Rightarrow S = \frac{1}{\sqrt{13}} \]

So, \( \hat{u} = \pm \frac{1}{\sqrt{13}} \langle -1, -1, 1 \rangle \)