Office hrs: Th, 11-12 & 3-4

**Directional Derivatives**

**Definition:** Let \( \vec{u} = \langle u_1, u_2 \rangle \) be a unit vector. Then

\[
D_{\vec{u}} f(a, b) := \lim_{h \to 0} \frac{f(a + u_1 h, b + u_2 h) - f(a, b)}{h}
\]

is the directional derivative of \( f \) at \((a, b)\) in the direction of \( \vec{u} \), (provided the limit above exists).

**Remarks:**

1. \( \langle a + u_1 h, b + u_2 h \rangle = \langle a, b \rangle + h \vec{u} \)
2. **Special cases:** \( \vec{u} = \vec{i} = \langle 1, 0 \rangle \) above
   \[ D_{\vec{i}} f(a, b) = f_x(a, b) \]
   Similarly, \[ D_{\vec{j}} f(a, b) = f_y(a, b) \]
   where \( \vec{j} = \langle 0, 1 \rangle \).

**Q:** How do we find the rate of change in \( f \) at \((a, b)\) as \((a, b)\) changes in the direction of some unit vector \( \vec{u} \)?
How do we calculate $D_{\vec{u}} f$?

**Remark**: $\vec{u}$ has to be a unit vector.

**Formula**: For $\vec{u} = \langle u_1, u_2 \rangle$,

$$D_{\vec{u}} f(a, b) = f_x(a, b) u_1 + f_y(a, b) u_2$$

Why is this formula working?

Let $L: \begin{cases} x = a + u_1 t \\ y = b + u_2 t \end{cases}$

$$D_{\vec{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

**Exercise**: verify $\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \bigg|_{t=0}$.

$$D_{\vec{u}} f(a, b) = f_x(x(0), y(0)) \cdot x'(0) + f_y(x(0), y(0)) \cdot y'(0)$$

Noting: $x(0) = a$, $y(0) = b$

$x'(0) = u_1$, $y'(0) = u_2$

we get

$$D_{\vec{u}} f(a, b) = u_1 f_x(a, b) \cdot u_1 + f_y(a, b) u_2$$
Ex: Suppose that you stand on the graph of \( f(x,y) \) over the origin.

Given: slope of the ground at \( P \)
- in \( N \)-direction = \(-5 \)
- in \( W \)-direction = \(-1 \)

(a) Slope of the ground in the direction of \( \langle 1,2 \rangle \)?
(b) Is there a direction with slope = 0? Find it.
(c) Find the direction of the greatest slope? (steepest upward)

Sol: (a) Since \( \langle 1,2 \rangle \) is not a unit vector, set \( \vec{u} = \frac{\langle 1,2 \rangle}{\| \langle 1,2 \rangle \|} = \frac{1}{\sqrt{5}} \langle 1,2 \rangle \) = \( \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle \)

Then \( \nabla f(0,0) = f_x(0,0) \cdot \frac{1}{15} + f_y(0,0) \cdot \frac{2}{15} \)

But \( f_x(0,0) = -5 \) \( f_y(0,0) = -1 \)

\( \Rightarrow \nabla f(0,0) = -5 \cdot \frac{1}{15} - \frac{2}{15} = -\frac{7}{15} \)

(b) \( \vec{u} = \langle u_1, u_2 \rangle \Rightarrow \)

\( \nabla f(0,0) = -5u_1 - u_2 = 0 \)

\( \Rightarrow \begin{align*}
   u_2 &= -5u_1 \\
   u_1^2 + u_2^2 &= 1
\end{align*} \)

\( \Rightarrow \) makes \( \vec{u} \) a unit vector

\( u_1 = \frac{1}{\sqrt{26}}, u_2 = \frac{-5}{\sqrt{26}} \)

or \( u_1 = -\frac{1}{\sqrt{26}}, u_2 = \frac{5}{\sqrt{26}} \)

\( \vec{u} = \langle \frac{1}{\sqrt{26}}, \frac{-5}{\sqrt{26}} \rangle \) or \( \vec{u} = \langle -\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \rangle \)
(c) **Goal:** maximize $D_{\mathbf{u}} f(0,0)$

over all unit vectors $\mathbf{u} = \langle u_1, u_2 \rangle$

which is:

$D_{\mathbf{u}} f(0,0) = -5 \cdot u_1 - u_2$

$= \langle -5, -1 \rangle \cdot \langle u_1, u_2 \rangle$

$= \| \langle -5, -1 \rangle \| \cdot \| \langle u_1, u_2 \rangle \| \cos \theta$

* $\theta$: angle between $\langle -5, -1 \rangle$ and $\mathbf{u}$

* $\| \mathbf{u} \| = \| \langle u_1, u_2 \rangle \| = 1$

Then:

$D_{\mathbf{u}} f(0,0) = \sqrt{26} \cdot 1 \cdot \cos \theta$

* This is maximized when $\theta = 0$

  that is: $\mathbf{u} = \frac{\langle -5, -1 \rangle}{\| \langle -5, -1 \rangle \|} = \frac{1}{\sqrt{26}} \langle -5, -1 \rangle$

  is the direction of greatest slope

So, the direction of greatest slope is the same as $\langle f_x(0,0), f_y(0,0) \rangle = \langle -5, -1 \rangle$.