[6] 1. Provide a sketch of: a plane with a normal vector \( \vec{n} \), a point \( Q \) on the plane, and a point \( P \) not on the plane.

(to simplify the rest of the problem, DO NOT make \( Q \) the closest point on the plane to \( P \)).

a) (2 marks) On your diagram, sketch and label the vectors \( \text{proj}_n \vec{QP} \) and \( \text{orth}_n \vec{QP} \).

b) (2 marks) Write an explicit formula for the distance from \( P \) to the plane in terms of the coordinates of the points \( P = (p_1, p_2, p_3) \), \( Q = (q_1, q_2, q_3) \) and \( \vec{n} = \langle a, b, c \rangle \) (you can leave your answer in terms of dot products and lengths of vectors).

c) (2 marks) Find the \( x \) coordinate of the point \( R \) on the plane which is closest to \( P \) in terms of those components given above.

\[
\overrightarrow{OR} = \overrightarrow{OP} - \text{proj}_n \vec{QP}
\]

\[\text{coord: } p_1 = \left\langle \frac{(p_1 - q_1, p_2 - q_2, p_3 - q_3) \cdot \langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}} \right\rangle_a \]
2. Consider the planes with equations \( x + 2y + 3z = 1 \) and \( x + y + z = 0 \).

a) (3 marks) Find the parametric equations of the line of intersection of these planes.

b) (3 marks) Find equations of three different planes (not including those above), which intersect at this line (hint: symmetric equations).

\[ \begin{align*}
\text{a)} & \quad x + 2y + 3z = 1 \\
& \quad x + y + z = 0 \\
\Rightarrow & \quad y + 3z = 1 \\
\Rightarrow & \quad y = 1 - 3z
\end{align*} \]

\[ \begin{cases}
z = t \\
y = 1 - 2t \\
x = -1 + t
\end{cases} \]

\[ \text{b)} \text{ Symmetric equations:} \]

\[ z = \frac{1 - y}{2} = x + 1 \]

Planes are:

\[ \begin{align*}
2z &= 1 - y \\
z &= x + 1 \\
2(x + 1) &= 1 - y
\end{align*} \]
3. Consider the vectors \( \vec{v} = <2, 2, 2>, \vec{w} = <0, -\sqrt{6}, \sqrt{6}> \) in space.

(a) (2 marks) Show that \( \vec{v}, \vec{w} \) are orthogonal and have equal length.

(b) (2 marks) When \( \vec{v}, \vec{w} \) are based at the origin they form two sides of a square \( S \). Find the coordinates of the four vertices of \( S \).

(c) (3 marks) Find an equation for the plane containing \( S \).

(d) (3 marks) Suppose \( S \) is the base of a rectangular box, and suppose the highest corner of the box just touches the plane \( z = 100 \) at a point \( P \). Find the coordinates of \( P \).

\[
\begin{align*}
\text{a)} & \quad \vec{v} \cdot \vec{w} = -2\sqrt{6} + 2\sqrt{6} = 0 \\
\text{b)} & \quad \text{corners are } (0, 0, 0), (2, 2, 2), (0, -\sqrt{6}, \sqrt{6}), (2, 2-\sqrt{6}, 2+\sqrt{6}) \\
\text{c}) & \quad \text{point } (0, 0, 0) \quad \text{normal: } \begin{vmatrix} i & j & k \\ 2 & 2 & 2 \\ 0 & -\sqrt{6} & \sqrt{6} \end{vmatrix} \\
& \quad = \langle 4\sqrt{6}, -2\sqrt{6}, -2\sqrt{6} \rangle \\
& \quad = 2\sqrt{6} \langle 2, -1, -1 \rangle \\
\text{d}) & \quad \text{highest pt on box is } (2, 2-\sqrt{6}, 2+\sqrt{6}) + t \langle 2, -1, -1 \rangle \\
& \quad \text{where } 2\sqrt{6} - t = 100 \\
& \quad \therefore t = \sqrt{6} - 98 \\
& \quad \therefore \text{pt is } (2, 2-\sqrt{6}, 2+\sqrt{6}) + (\sqrt{6} - 98)(2, -1, -1)
\end{align*}
\]
4. 

a) (4 marks) Sketch the quadric surface $x + y^2 + 4z^2 = 0$.

b) (2 marks) Sketch the hyperbola $z^2 - \frac{x^2}{4} = 1$ in the $x,z$ plane. Now sketch the surface in space obtained by rotating this hyperbola around the $z$ axis.

c) (2 marks) The surface in part (b) is a quadric surface in space. What is its equation in $x,y,z$?

\[ \frac{x^2}{4} - \frac{z^2}{1} = 1 \]
[10] 5. Consider the graph of the function \( f(x, y) = \sqrt{\frac{y}{2 + \sin x}} + 1 \).

a) (4 marks) Sketch the domain and state the range of \( f(x, y) \) (you do not need to prove its range, just state it).

b) (4 marks) Sketch the \( k \)-level curve of \( f(x, y) \) for \( k = 2 \).

c) (2 marks) Suppose you are hiking on the surface \( z = f(x, y) \) in space. You want to hike along a path which is at a constant height, and also as straight as possible. Find such a path on the surface.

\[
\frac{y}{2 + \sin x} + 1 \geq 0 \quad \Rightarrow \quad y \geq -2 - \sin x
\]

\[
f(x, y) = 2 \quad \Rightarrow \quad y = (2^{2-1}) (2 + \sin x)
= 6 + 3 \sin x
\]

\[
\text{when } k = 1, \text{ level curve is } y = (1 - 1)(2 + \sin x) = 0 \!
\]

a) (2 marks) Find the first partial derivatives of the function \( f(x, y) = e^{xy^2} \)

\[
\begin{align*}
\frac{f}{x} &= e^{xy^2} \cdot y^2 \\
\frac{f}{y} &= e^{xy^2} \cdot 2xy
\end{align*}
\]

b) (4 marks) Consider the surface \( z = x^2 + 2x - 2xy - y^2 \) in space. You want to place a ball on the surface so that it stays in one place (does not roll off). Over which point in the \( x, y \) plane should you place the ball?

\[
\begin{align*}
\frac{z}{x} &= 2x + 2 - 2y > 0 \\
\frac{z}{y} &= -2x - 2y = 0
\end{align*}
\]

\[\Rightarrow 2 - 4y = 0 \]

\[\Rightarrow y = \frac{1}{2}, \quad x = -\frac{1}{2}\]

c) (4 marks) For what value for the constant \( k \) does the 2 variable function \( u(x, t) = e^{kt} \sin(3x) \) satisfy the partial differential equation \( u_t = 2u_{xx} \)?

\[
\begin{align*}
u_t &= k e^{kt} \cos(3x) \\
u_x &= e^{kt} \cos(3x) \cdot 3 \\
u_{xx} &= e^{kt} (-\sin(3x)) \cdot 9
\end{align*}
\]

\[\therefore u_t = 2u_{xx} \quad \Rightarrow \quad k = 2(-9) = -18 \]

The End