Problem 1: All parts of this problem refer to four points $A$, $B$, $C$, $D$, where $A(0,0,0)$ is the origin, and the coordinates of the other three are: $B(0,1,0)$, $C(1,0,1)$, $D(3,0,-3)$.

(a) [2 points] Find the component of the vector $\overrightarrow{BC}$ along the vector $\overrightarrow{BD}$.

(b) [2 points] Find the area of the triangle $ABC$.

(c) [3 points] Find the equation of the plane containing the points $A$, $B$, and $C$.

(d) [3 points] Find the distance from the point $D$ to the plane containing $A$, $B$, and $C$.

(e) [3 points] Find the volume of the parallelepiped spanned by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$.

(f) [4 points] Find the parametric equation for the line of intersection of the plane $x = z$ and the plane $x + 2y + z = 1$.

**Solutions**

(a) \[
\overrightarrow{BC} = \langle 1, -1, 1 \rangle,
\overrightarrow{BD} = \langle 3, -1, 3 \rangle,
\text{comp}_{\overrightarrow{BD}} \overrightarrow{BC} = \frac{\overrightarrow{BC} \cdot \overrightarrow{BD}}{|\overrightarrow{BD}|} = \frac{\langle 1, -1, 1 \rangle \cdot \langle 3, -1, 3 \rangle}{\sqrt{9 + 1 + 9}} = \frac{1}{\sqrt{19}}
\]

(b) \[
\overrightarrow{AB} = \langle 0, 1, 0 \rangle,
\overrightarrow{AC} = \langle 1, 0, 1 \rangle,
\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \vec{E} - \vec{E},
|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} |\vec{E} - \vec{E}| = \frac{1}{2} \sqrt{1 + 1} = \left\lfloor \frac{1}{2} \sqrt{2} \right\rfloor
\]

(c) A normal vector to this plane is $\overrightarrow{AB} \times \overrightarrow{AC}$, which we just found in (b) — it is $\overrightarrow{E} - \overrightarrow{E} = \langle 1, 0, -1 \rangle$.

We can use the point $A$ as "the point on the plane".

*Answer: $x - 2 = 0$*

(d) Distance from $D$ to the plane is the absolute value of the component of the vector $\overrightarrow{AD}$ along $\vec{n} = \langle 1, 0, -1 \rangle$.

*Answer: $|\text{comp}_{\vec{n}} \overrightarrow{AD}| = \left| \frac{\langle 1, 0, -1 \rangle \cdot \langle 3, 0, -3 \rangle}{\sqrt{2}} \right| = \left\lfloor \frac{6}{\sqrt{2}} \right\rfloor$*
Math 200 Midterm I (October 11, 2012)
Sections 107. Instructor: Julia Gordon

**Problem 1:** All parts of this problem refer to four points \( A, B, C, D \), where \( A(0,0,0) \) is the origin, and the coordinates of the other three are: \( B(0,1,0), C(1,0,1), D(3,0,-3) \).

(a) \([2 \text{ points}]\) Find the component of the vector \( \overrightarrow{BC} \) along the vector \( \overrightarrow{BD} \).

(b) \([2 \text{ points}]\) Find the area of the triangle \( ABC \).

(c) \([3 \text{ points}]\) Find the equation of the plane containing the points \( A, B, \) and \( C \).

(d) \([3 \text{ points}]\) Find the distance from the point \( D \) to the plane containing \( A, B, \) and \( C \).

(e) \([3 \text{ points}]\) Find the volume of the parallelepiped spanned by the vectors \( \overrightarrow{AB}, \overrightarrow{AC}, \) and \( \overrightarrow{AD} \).

(f) \([4 \text{ points}]\) Find the parametric equation for the line of intersection of the plane \( x = z \) and the plane \( x + 2y + z = 1 \).

\[ \text{(e)}: \text{can use the scalar triple product:} \]
\[ V = \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & -3 \end{vmatrix} = 6 \]

Another way is to notice that this volume equals the distance from \( D \) to the plane \( (AB,C) \) times the area of the parallelogram spanned by \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \), which is twice the area of the triangle from (b).

So, we get
\[ V = 2 \cdot \text{(answer in (b))} \cdot \text{(answer in (d))} \]
\[ = 2 \cdot \frac{1}{2} \sqrt{2} \cdot \frac{6}{\sqrt{2}} = 6 \]
Problem 1: All parts of this problem refer to four points \( A, B, C, D \), where \( A(0,0,0) \) is the origin, and the coordinates of the other three are: \( B(0,1,0), C(1,0,1), D(3,0,-3) \).

(a) \[2 \text{ points}\] Find the component of the vector \( \overrightarrow{BC} \) along the vector \( \overrightarrow{BD} \).

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(f) The direction vector of this line is parallel to both planes, so we can obtain it by taking the cross product of the normal vectors.

So, \( \overrightarrow{v} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 2i - 2j + 2k \)

Common point: \( \begin{cases} x = 2 \\ x + 2y + z = 1 \end{cases} \)

Can take \( x = 2 = 0 \), then \( y = \frac{1}{2} \)

Answer: \( \begin{cases} x(t) = 2t \\ y(t) = \frac{1}{2} - 2t \\ z(t) = 2t \end{cases} \)

(There are many equivalent answers).
Problem 2:

(a) [3 points] Find the parametric equation of the line through the points $(1, 1, 2)$ and $(0, 1, 3)$.

Then \[ \begin{align*}
  x(t) & = 1 - t \\
  y(t) & = 1 \\
  z(t) & = 2 + t
\end{align*} \]

(b) [3 points] Find the symmetric equation of the same line.

(c) [3 points] Find the angle between this line and the plane $y = x$.

(d) [3 points] Find the point of intersection of your line with the plane $3x + y + 2z = 6$.

\[ \text{Vector from } (1, 1, 2) \text{ to } (0, 1, 3) = \langle -1, 0, 1 \rangle \]

Then \[ \begin{align*}
  x(t) & = 1 - t \\
  y(t) & = 1 \\
  z(t) & = 2 + t
\end{align*} \]

(There are many equivalent correct answers)

\[ y = 1, \quad x - 1 = z - 2 \]

(c) First, find the angle $\beta$ between the line and the normal vector to the plane. (Remember, we want an acute angle.) The normal vector is $\mathbf{n} = \langle 1, -1, 0 \rangle$.

\[
\cos \beta = \left| \frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{v}| |\mathbf{n}|} \right| = \left| \frac{\langle 1, 0, 1 \rangle \cdot \langle 1, -1, 0 \rangle}{\sqrt{2} \sqrt{2}} \right| = \frac{1}{2}
\]

Then \[ \beta = \cos^{-1} \left( \frac{1}{2} \right) = \frac{\pi}{3} \]

Now, the angle $\alpha$ that we want is \[ \frac{\pi}{2} - \beta = \frac{\pi}{6} \]

Answer: \[ \alpha = \frac{\pi}{6} \]

(d) we need the point $(x(t), y(t), z(t))$ on the line to satisfy the equation of the plane. Plug it in and solve for $t$:

\[ 3(1 - t) + 1 + 2(2 + t) = 6 \]
\[ 3 - 3t + 1 + 4 + 2t = 6 \]
\[ 8 - t = 6 \]
\[ t = 2 \]

Then \[ \langle x(t), y(t), z(t) \rangle = \left( \frac{1 - 2}{1 - t}, \frac{1}{1 - t}, \frac{2 + 2}{2 + t} \right) = \left( -1, 1, \frac{4}{4} \right) \]
Problem 3: All parts of this problem are about the surface \( z = 9 - x^2 - 4y^2 \).

(a) [3 points] Sketch the traces of this surface on the planes \( z = 0 \), and \( x = 1 \). Label the axes in your plots, and label as many features of the plots as possible.

(b) [2 points] Classify this surface and sketch it.

(c) [4 points] Find the equation of the tangent plane to this surface at the point \((1, 1, 4)\).

(d) [4 points] Find the point \((a, b, c)\) on the surface such that the tangent plane at \((a, b, c)\) is parallel to the plane \( z = -x - 3y \).

\[ z = 9 - x^2 - 4y^2 = f(x, y) \]

The tangent plane has the equation

\[ z = L(x, y) \]

where \( L(x, y) \) is the linearization of \( f(x, y) \) at the point \((1, 1)\).

\[ f_x = -2x \quad f_x(1, 1) = -2 \]
\[ f_y = -8y \quad f_y(1, 1) = -8 \]

Then \( L(x, y) = 4 - 2(x-1) - 8(y-1) \), and the answer is

\[ z = 4 - 2(x-1) - 8(y-1) \]
In general, if \((a,b,c)\) is a point on our paraboloid, then \(c = 9 - a^2 - 4b^2\). Then we need to find \(a\) and \(b\).

The tangent plane at \((a,b,c)\) will have the equation

\[
\tilde{z} = c + f_x(a,b)(x-a) + f_y(a,b)(y-b).
\]

Then the normal vector to the tangent plane is \(\langle f_x(a,b), f_y(a,b), -1 \rangle\).

We need it to be proportional to the vector \(\langle -1, -3, -1 \rangle\) (the normal of the given plane).

So, we have

\[
\begin{align*}
    f_x(a,b) &= k \cdot (-1) \\
    f_y(a,b) &= k \cdot (-3)
\end{align*}
\]

for some \(k\).

Then \(k = 1\). So we need to find \(a, b\) such that

\[
\begin{align*}
    f_x(a,b) &= -1 \\
    f_y(a,b) &= -3
\end{align*}
\]

We know:

\[
\begin{align*}
    f_x(a,b) &= -2a \\
    f_y(a,b) &= -8b
\end{align*}
\]

Then

\[
\begin{align*}
    -2a &= -1 \\
    -8b &= -3
\end{align*}
\]

So,

\[
a = \frac{1}{2}, \quad b = \frac{3}{8}, \quad c = 9 - \left(\frac{1}{2}\right)^2 - 4 \cdot \left(\frac{3}{8}\right)^2
\]
Problem 4:

(a) [2 points] Let \( g(x, y) = e^x \cos y - y^2 x \). Find \( \frac{\partial g}{\partial y} \).

(b) [3 points] Is there a function \( f(x, y) \) that satisfies:

\[
\begin{align*}
    f_x &= e^x \cos y - y^2 x \\
    f_y &= e^x \sin y + x^2 y.
\end{align*}
\]

Explain your reasoning.

(a) \[ \frac{\partial g}{\partial y} = -e^x \sin y - 2xy \]

(b) Let us compare \( \frac{\partial}{\partial y}(e^x \cos y - y^2 x) \)

and \( \frac{\partial}{\partial x}(e^x \sin y + x^2 y) \).

By Clairaut's Theorem, for any function with continuous second partial derivatives, we must have \( f_{xy} = f_{yx} \). So if such a \( f \) existed, these expressions have to be the same.

We have:

\[ \frac{\partial}{\partial y}(e^x \cos y - y^2 x) = -e^x \sin y - 2xy \]

\[ \frac{\partial}{\partial x}(e^x \sin y + x^2 y) = e^x \sin y + 2xy \]

and these two are not equal (but are continuous).

Thus \( f \) cannot exist.