3. \[ |e^{i\theta}| = |\cos \theta + i \sin \theta| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1. \]

For all \( \theta \) real!

\[ e^{i \theta} \]

Unit circle

Ex: Find all real solutions of

\[ e^{i(\theta + \theta^2)} = 4. \]

Sol: No such \( \theta \)!

4. \[ e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)} \]

Then:

\[ z_1 = r_1 \cdot e^{i\theta_1} \quad (r_1 = |z_1|) \]

\[ z_2 = r_2 \cdot e^{i\theta_2} \]

\[ z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i(\theta_1 + \theta_2)} \]

\[ z_1 / z_2 = r_1 / r_2 \cdot e^{i(\theta_1 - \theta_2)} \]

Multiplication/division is very easy in this form!

Last time:

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

For any \( z = a + ib \), we have

\[ z = r \cdot e^{i\theta} \quad \text{where} \quad r = |z| \]

\[ \theta = \text{Arg}(z). \]
Ex: \( z_1 = 1 + i = \sqrt{2} \cdot e^{i\pi/4} \)

\( z_2 = 1 - i = \sqrt{2} \cdot e^{-i\pi/4} \)

\( z_1 \cdot z_2 = \sqrt{2} \cdot \sqrt{2} \cdot e^{i(\pi/4 - \pi/4)} = 2 \)

\( \frac{z_1}{z_2} = \frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}} = e^{i\pi/2} = i \)

\( z_1^5 = (\sqrt{2})^5 \cdot e^{5i\pi/4} \)

Roots: Find \( z \) s.t. \( z^3 = 1 \).

(i.e., find all roots of \( P(z) = z^3 - 1 \).)

Clearly, one root is \( z_1 = 1 \).

Using \( z_1 = 1 \): \( z^3 - 1 = (z-1)(z^2 + z + 1) \)

Now, go back to \( z^3 = 1 \)

Suppose \( z = r e^{i\theta} \rightarrow \)

\( r^3 e^{i3\theta} = 1 \), \( r \in \mathbb{R} \)

\( \theta \in \mathbb{R} \)

\( \Rightarrow r^3 e^{i3\theta} = 1 \), \( e^{i(0 + 2k\pi)} = 1 \)
Then: \[ r = 1 \]
\[
3\theta = 0 \Rightarrow \theta = 0
\]
\[
3\theta = 2\pi \Rightarrow \theta = \frac{2\pi}{3}
\]
\[
3\theta = 4\pi \Rightarrow \theta = \frac{4\pi}{3}
\]

Conclusion: The roots of \( z^3 = 1 \) are:
\[
z_1 = 1 \cdot e^{i0} = 1
\]
\[
z_2 = 1 \cdot e^{i\frac{2\pi}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i
\]
\[
z_3 = 1 \cdot e^{i\frac{4\pi}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i
\]

Then:
\[
z^3 - 1 = (z - 1)(z - (-\frac{1}{2} + \frac{\sqrt{3}}{2}i))(z - (-\frac{1}{2} - \frac{\sqrt{3}}{2}i))
\]

Ex: \( z^5 = -3 \). Find all the roots.

Sol: \( r = 3 \cdot e^{i\frac{\pi}{2}} = -3 \)

Then: \( r = 3^{\frac{1}{5}} \)

\[
5\theta = \pi \Rightarrow \theta_1 = \frac{\pi}{5}
\]
\[
5\theta = 3\pi \Rightarrow \theta_2 = \frac{3\pi}{5}
\]
\[
5\theta = 5\pi \Rightarrow \theta_3 = \frac{5\pi}{5} = \pi
\]
\[
5\theta = 7\pi \Rightarrow \theta_4 = \frac{7\pi}{5}
\]
\[
5\theta = 9\pi \Rightarrow \theta_5 = \frac{9\pi}{5}
\]

Stop after 5 consequent ones

\[
5\theta = 11\pi \Rightarrow \theta_6 = \frac{11\pi}{5} = \frac{2\pi}{5}
\]
Then, e.g., \( z_1 = 3^{1/5} (\cos(\pi/5) + i \sin(\pi/5)) \)

**Eigenvalues / Eigenvectors**

Suppose we have an \( m \times m \) matrix \( A \) (square!) and we want to compute \( A^n x \) for any given \( x \) and any (possibly large) \( n \)!

Ex: \[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \]

First, observe that \( \text{rank}(A) = 2 \), so the homog. system \[ A x = \vec{0} \]

We know how to find it:

\[ x_n = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \]

That is: \( A x_n = \vec{0} \) for any \( s \)!

Let's pick \( s = 1 \) and set \( v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \). Then

\[ A v_1 = \vec{0} \]

With this \( v_1 \), we have

\[ A^2 v_1 = A(A v_1) = A \vec{0} = \vec{0} \]

\[ A^n v_1 = \vec{0} \quad \text{and have a formal for } A^n v_1! \]
So, $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is special in that multiplying $v_1$ by $A$ is like multiplying $v_1$ by the scalar 0.

Another special vector:

$v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ (I'm just giving this to you - will learn how to find it soon)

$A v_2 = \begin{bmatrix} 4 \\ 6 \\ 4 \\ 4 \end{bmatrix}$

That is:

$A v_2 = v_2 \iff$ multiplying by $A$ is like multiplying by the scalar 1

Then $A^n v_2 = v_2$

Finally:

$v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$A v_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \end{bmatrix} = 2 v_3$

That is: $A v_3 = 2 v_3$, so multiplying $v_3$ by $A$ is like multiplying $v_3$ by the scalar 2!
Note that \( \{v_1, v_2, v_3\} \) are linearly independent in \( \mathbb{R}^3 \), so \( \{v_1, v_2, v_3\} \) is a basis for \( \mathbb{R}^3 \).

That is:

\[
\forall x \in \mathbb{R}^3, \text{ there exist } c_1, c_2, c_3 \in \mathbb{R} \text{ s.t. } x = c_1 v_1 + c_2 v_2 + c_3 v_3
\]

Then

\[
A x = A (c_1 v_1 + c_2 v_2 + c_3 v_3)
= c_1 A v_1 + c_2 A v_2 + c_3 A v_3
\]

\[
= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + c_3 \lambda_3 v_3
\]

\[
A^2 x = c_2 \lambda_2 v_2 + 2 c_3 \lambda_3 v_3
= c_2 v_2 + 2^2 c_3 v_3
\]

So, if we know (like in this example) the "special vectors" \( v_1, v_2, v_3 \) s.t.

\[
A v_1 = \lambda_1 v_1
A v_2 = \lambda_2 v_2
A v_3 = \lambda_3 v_3
\]

such that \( \{v_1, v_2, v_3\} \) is a basis for \( \mathbb{R}^3 \), then

\[
A^n x = c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2 + c_3 \lambda_3^n v_3
\]

where \( x = c_1 v_1 + c_2 v_2 + c_3 v_3 \).

Same remark above generalizes to an \( m \times m \) matrix \( A \) — in that case we have \( \lambda_1, \lambda_2, \ldots, \lambda_m \) (in eigenvalues) possibly complex.