Midterm Exam
Duration: 80 minutes

• This midterm has 4 questions, each worth 20 points, for a total of 80 points.

• This is a closed-book examination. None of the following is allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

• Please write your name on every page.
Question 1 (20 pts.)

1.1. Let \( \ell \) be a line which is parallel to the plane

\[
2x + y - z - 5 = 0
\]

and perpendicular to the line

\[
x = 3 - t \quad y = 1 - 2t, \quad z = 3t.
\]

Find a vector parallel to the line \( \ell \).

**Answer:** Normal to given plane is

\[
\vec{n} = \langle 2, 1, -1 \rangle.
\]

Direction vector of the given line is

\[
\vec{d} = \langle -1, -2, 3 \rangle.
\]

The line \( \ell \) is perpendicular to both \( \vec{n} \) and \( \vec{d} \), so the answer is

\[
\vec{n} \times \vec{d} = \langle 1, -5, -3 \rangle
\]

(or any vector which is this multiplied by a nonzero scalar).
1.2. A **tetrahedron** is a solid with four vertices $P$, $Q$, $R$, and $S$, and four triangular faces, as shown in the figure.

![Diagram of a tetrahedron]

The volume of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face. Find the volume of the tetrahedron whose vertices are $P = (1, 1, 1)$, $Q = (1, 2, 3)$, $R = (1, 1, 2)$, and $S = (3, -1, 2)$.

**Answer:** We have

\[
PQ = \langle 0, 1, 2 \rangle \\
PR = \langle 0, 0, 1 \rangle \\
PQ \times PR = \langle 1, 0, 0 \rangle \\
\|PQ \times PR\| = 1
\]

So

\[
\text{Area(\Delta PQR)} = \frac{1}{2} \|PQ \times PR\| = \frac{1}{2}.
\]

The plane that supports the face $PQR$ has normal

\[
\vec{n} = PQ \times PR = \langle 1, 0, 0 \rangle
\]

and it goes through the point $P$, so it is given by

\[
\langle x - 1, y - 1, z - 1 \rangle \cdot \langle 1, 0, 0 \rangle = 0
\]

which gives the equation

\[
x - 1 = 0.
\]

Distance between the point $S$ and the plane, by formula learned in class, is

\[
\frac{|1 \cdot 3 + 0 \cdot (-1) + 0 \cdot 2 - 1|}{\sqrt{1^2 + 0^2 + 0^2}} = 2.
\]

So the answer is

\[
\text{Vol}(PQRS) = \frac{1}{3} \cdot 2 \cdot \frac{1}{2} = \frac{1}{3}.
\]
Question 2 (20 pts.)

Consider the surface given by
\[ z = 6 - x - x^2 - 2y^2. \]

2.1. Describe/sketch the traces of this surface for \( y = 2 \) and \( z = 3 \).

Answer:
\( y = 2 \) is parabola and \( z = 3 \) is an ellipse.
2.2. Find a tangent plane to this surface at the point \((1, 2, -4)\).

**Answer:** Define \(f(x, y) = 6 - x - x^2 - 2y^2\). Then the surface in the question is the graph of \(f\) and the tangent plane is given by the equation
\[
z = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2).
\]

We have
\[
f_x(x, y) = -1 - 2x \quad f_x(1, 2) = -1 - 2 \cdot 1 = -3
\]
\[
f_y(x, y) = -4y \quad f_y(x, y) = -4 \cdot 2 = -8
\]
and of course \(f(1, 2) = -4\). So the answer is
\[
z = -4 - 3(x - 1) - 8(y - 2)
\]
or
\[
z = -3x - 8y + 15.
\]

2.3. The intersection of the surface with the plane \(x = 1\) is a parabola. Find parametric equations for the line tangent to this parabola at the point \((1, 2, -4)\).

[Hint: This line lies in both the tangent plane from the previous question 2.2 and in the plane \(x = 1\).]

**Answer:**
The line we’re interested in lies on both planes \(z = -3x - 8y + 15\) and \(x = 1\). So its direction vector is perpendicular to both normals: \((-3, -8, -1)\) and \((1, 0, 0)\). Thus a direction vector for this line can be taken to be
\[
\vec{d} = \langle 1, 0, 0 \rangle \times \langle -3, -8, -1 \rangle = \langle 0, 1, -8 \rangle.
\]

Since the tangent line also goes through the point \((1, 2, -4)\), we get
\[
x = 1 \quad y = 2 + t \quad z = -4 - 8t.
\]
2.4. Describe/sketch the level curves of the function

\[ g(x, y) = \frac{e^{x^2 + y^2}}{2 + \cos(1 + x^2 + y^2)}. \]

**Answer:**

The level curves are circles centered at the origin.

As an example, consider pairs \((x, y)\) such that \(x^2 + y^2 = 1\) (that is, pairs \((x, y)\) that lie on the circle of radius 1 centered at the origin). For such pairs \((x, y)\) we have

\[ g(x, y) = \frac{e^{x^2 + y^2}}{2 + \cos(1 + x^2 + y^2)} = \frac{e^1}{2 + \cos(1 + 1)} = \frac{e}{1 + \cos(2)}. \]

This means that the level curve that corresponds to \(z = \frac{e}{1 + \cos(2)}\) is the circle \(x^2 + y^2 = 1\).

In general, the level curve \(z = \frac{e^{r^2}}{2 + \cos(1 + r^2)}\) is the circle \(x^2 + y^2 = r^2\).
**Question 3 (20 pts.)**
Consider the function \( f(x, y) = 3y^2 - 2x^2 + x. \)

3.1. Find \( f_x, f_y, f_{xx}, f_{yx}. \)

**Answer:**

\[
\begin{align*}
  f_x(x, y) &= -4x + 1 \\
  f_y(x, y) &= 6y \\
  f_{xx}(x, y) &= -4 \\
  f_{yx}(x, y) &= 0
\end{align*}
\]

3.2. Find an equation of the tangent plane \( h \) to the graph of \( f \) at the point \((a, b, f(a, b))\), where \(a, b\) are some parameters.

**Answer:**

\[
\begin{align*}
  z &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\
  z &= 3b^2 - 2a^2 + a + (-4a + 1)(x - a) + 6b(y - b)
\end{align*}
\]
3.3. Find parametric equations for the line $\ell$ that goes through the point $P = (a, b, f(a, b))$ and the point $Q = (1, 1, 1)$.

**Answer:**

$$\ell(t) = (1, 1, 1) + t(1 - a, 1 - b, 1 - f(a, b))$$

3.4. For what pairs $(a, b)$ does the line $\ell$ lie in the plane $h$?

Find an equation that describes all such pairs (this equation should involve only $a$ and $b$) and describe/sketch the curve in the plane determined by this equation.

**Answer:**

*One possible solution:*

Since $\ell$ intersects $h$ (at the point $(a, b, f(a, b))$) it suffices to require that it will also be parallel to $h$.

A normal vector to $h$ is

$$\langle -4a + 1, 6b, -1 \rangle$$

and the direction of $\ell$ is

$$\langle 1 - a, 1 - b, 1 - f(a, b) \rangle.$$

So we want

$$\langle -4a + 1, 6b, -1 \rangle \cdot \langle 1 - a, 1 - b, 1 - f(a, b) \rangle = 0.$$

This gives

$$2a^2 - 4a - 3b^2 + 6b = 0$$

which can be rewritten as

$$2(a - 1)^2 - 3(b - 1)^2 = -1$$

or

$$\frac{(b - 1)^2}{1/3} - \frac{(a - 1)^2}{1/2} = 1$$

which is a hyperbola.
**Question 4 (20 pts.)**
Consider an isosceles triangle with edge lengths \(a, a,\) and \(c,\) and with an angle \(\theta,\) as shown in the figure.

![Isosceles Triangle Diagram](image)

The law of cosines, applied to such isosceles triangles, states

\[
c^2 = 2a^2(1 - \cos \theta).
\]

4.1. In view of this equation, we may regard \(\theta\) as a function \(\theta = \theta(a,c)\) of \(a\) and \(c.\)

What is \(\theta(1, \sqrt{2})?\)

**Answer:**

\[
\theta(1, \sqrt{2}) = \frac{\pi}{2}
\]

4.2. Using implicit differentiation, find \(\frac{\partial \theta}{\partial a}\) and \(\frac{\partial \theta}{\partial c}.\)

**Answer:**

*To find \(\frac{\partial \theta}{\partial a}:\)*

\[
\frac{\partial}{\partial a} (c^2) = \frac{\partial}{\partial a} (2a^2(1 - \cos \theta))
\]

\[
0 = 4a(1 - \cos \theta) + 2a^2 \cdot \sin \theta \cdot \frac{\partial \theta}{\partial a}
\]

so

\[
\frac{\partial \theta}{\partial a} = \frac{-2(1 - \cos \theta)}{a \sin \theta}.
\]

*To find \(\frac{\partial \theta}{\partial c}:\)*

\[
\frac{\partial}{\partial c} (c^2) = \frac{\partial}{\partial c} (2a^2(1 - \cos \theta))
\]

\[
2c = 2a^2 \cdot \sin \theta \cdot \frac{\partial \theta}{\partial c}
\]

so

\[
\frac{\partial \theta}{\partial c} = \frac{-c}{a^2 \sin \theta}.
\]
4.3. Now regard \( c \) as a function \( c = c(a, \theta) \) of \( a \) and \( \theta \). What is \( c(1, \frac{\pi}{3}) \)? (Recall that \( \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \).)

**Answer:**

\[
c(1, \frac{\pi}{3}) = 1
\]

4.4. Find linear approximation of \( c = c(a, \theta) \) at the point \( (1, \frac{\pi}{3}) \).

**Answer:**

\[
L(a, \theta) = c(a, \theta) + \frac{\partial c}{\partial a}(a, \theta) \cdot (a - 1) + \frac{\partial c}{\partial \theta}(a, \theta) \cdot (\theta - \frac{\pi}{3}).
\]

To compute \( \frac{\partial c}{\partial a} \) and \( \frac{\partial c}{\partial \theta} \) we use implicit differentiation, this time regarding \( a, \theta \) as independent variables and \( c \) as a function \( c(a, \theta) \).

To find \( \frac{\partial c}{\partial a} \):

\[
\frac{\partial}{\partial a} (c^2) = \frac{\partial}{\partial a} (2a^2(1 - \cos \theta))
\]

\[
2c \frac{\partial c}{\partial a} = 4a(1 - \cos \theta)
\]

\[
\frac{\partial c}{\partial a} = \frac{2a(1 - \cos \theta)}{c}
\]

and thus

\[
\frac{\partial c}{\partial a}(1, \frac{\pi}{3}) = 1.
\]

To find \( \frac{\partial c}{\partial \theta} \):

\[
\frac{\partial}{\partial \theta} (c^2) = \frac{\partial}{\partial \theta} (2a^2(1 - \cos \theta))
\]

\[
2c \frac{\partial c}{\partial \theta} = 2a^2 \sin \theta
\]

\[
\frac{\partial c}{\partial \theta} = \frac{a^2 \cdot \sin \theta}{c}
\]

and thus

\[
\frac{\partial c}{\partial \theta}(1, \frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}.
\]

Hence

\[
L(a, \theta) = 1 + 1 \cdot (a - 1) + \frac{\sqrt{3}}{2}(\theta - \frac{\pi}{3}).
\]