Midterm Exam
Duration: 80 minutes

• This midterm has 4 questions, each worth 20 points, for a total of 80 points.

• This is a closed-book examination. None of the following is allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

• Please write your name on every page.
Question 1 (20 pts.)

1.1. Let \( \ell \) be a line which is parallel to the plane

\[
2x + y - z - 5 = 0
\]

and perpendicular to the line

\[
x = 3 - t \quad y = 1 - 2t, \quad z = 3t.
\]

Find a vector parallel to the line \( \ell \).

**Answer:** Normal to given plane is

\[
\vec{n} = \langle 2, 1, -1 \rangle.
\]

Direction vector of the given line is

\[
\vec{d} = \langle -1, -2, 3 \rangle.
\]

The line \( \ell \) is perpendicular to both \( \vec{n} \) and \( \vec{d} \), so the answer is

\[
\vec{n} \times \vec{d} = \langle 1, -5, -3 \rangle
\]

(or any vector which is this multiplied by a nonzero scalar).
1.2. A tetrahedron is a solid with four vertices $P$, $Q$, $R$, and $S$, and four triangular faces, as shown in the figure.

The volume of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face. Find the volume of the tetrahedron whose vertices are $P = (1,1,1)$, $Q = (1,2,3)$, $R = (1,1,2)$, and $S = (3,-1,2)$.

**Answer:** We have

\[ PQ = \langle 0,1,2 \rangle \]
\[ PR = \langle 0,0,1 \rangle \]
\[ PQ \times PR = \langle 1,0,0 \rangle \]
\[ \|PQ \times PR\| = 1 \]

So
\[ \text{Area}(\Delta PQR) = \frac{1}{2} \|PQ \times PR\| = \frac{1}{2}. \]

The plane that supports the face $PQR$ has normal
\[ \vec{n} = PQ \times PR = \langle 1,0,0 \rangle \]
and it goes through the point $P$, so it is given by
\[ \langle x-1, y-1, z-1 \rangle \cdot \langle 1,0,0 \rangle = 0 \]
which gives the equation
\[ x-1 = 0. \]

Distance between the point $S$ and the plane, by formula learned in class, is
\[ \frac{|1 \cdot 3 + 0 \cdot (-1) + 0 \cdot 2 - 1|}{\sqrt{1^2 + 0^2 + 0^2}} = 2. \]

So the answer is
\[ \text{Vol}(PQRS) = \frac{1}{3} \cdot 2 \cdot \frac{1}{2} = \frac{1}{3}. \]
Question 2 (20 pts.)

Consider the surface given by
\[ z = 6 - x - x^2 - 2y^2. \]

2.1. Describe/sketch the traces of this surface for \( y = 2 \) and \( z = 3 \).

Answer:
\( y = 2 \) is parabola and \( z = 3 \) is an ellipse.
2.2. Find a tangent plane to this surface at the point $(1, 2, -4)$.

**Answer:** Define $f(x, y) = 6 - x - x^2 - 2y^2$. Then the surface in the question is the graph of $f$ and the tangent plane is given by the equation

$$z = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2).$$

We have

$$f_x(x, y) = -1 - 2x$$
$$f_x(1, 2) = -1 - 2 \cdot 1 = -3$$
$$f_y(x, y) = -4y$$
$$f_y(x, y) = -4 \cdot 2 = -8$$

and of course $f(1, 2) = -4$. So the answer is

$$z = -4 - 3(x - 1) - 8(y - 2)$$

or

$$z = -3x - 8y + 15.$$

2.3. The intersection of the surface with the plane $x = 1$ is a parabola. Find parametric equations for the line tangent to this parabola at the point $(1, 2, -4)$.

[Hint: This line lies in both the tangent plane from the previous question 2.2 and in the plane $x = 1$.]

**Answer:**

The line we’re interested in lies on both planes $z = -3x - 8y + 15$ and $x = 1$. So its direction vector is perpendicular to both normals: $\langle -3, -8, -1 \rangle$ and $\langle 1, 0, 0 \rangle$. Thus a direction vector for this line can be taken to be

$$\vec{d} = \langle 1, 0, 0 \rangle \times \langle -3, -8, -1 \rangle = \langle 0, 1, -8 \rangle.$$

Since the tangent line also goes through the point $(1, 2, -4)$, we get

$$x = 1, \quad y = 2 + t, \quad z = -4 - 8t.$$
2.4. Describe/sketch the level curves of the function
\[ g(x, y) = \frac{e^{x^2+y^2}}{2 + \cos(1 + x^2 + y^2)}. \]

**Answer:**
The level curves are circles centered at the origin.

As an example, consider pairs \((x, y)\) such that \(x^2 + y^2 = 1\) (that is, pairs \((x, y)\) that lie on the circle of radius 1 centered at the origin). For such pairs \((x, y)\) we have
\[ g(x, y) = \frac{e^{x^2+y^2}}{2 + \cos(1 + x^2 + y^2)} = \frac{e^1}{2 + \cos(1 + 1)} = \frac{e}{1 + \cos(2)}. \]

This means that the level curve that corresponds to \(z = \frac{e}{1 + \cos(2)}\) is the circle \(x^2 + y^2 = 1\).

In general, the level curve \(z = \frac{e^{r^2}}{2 + \cos(1 + r^2)}\) is the circle \(x^2 + y^2 = r^2\).
Question 3 (20 pts.)
Consider the function \( f(x, y) = 3y^2 - 2x^2 + x \).

3.1. Find \( f_x, f_y, f_{xx}, f_{yx} \).

Answer:

\[
\begin{align*}
  f_x(x, y) &= -4x + 1 \\
  f_y(x, y) &= 6y \\
  f_{xx}(x, y) &= -4 \\
  f_{yx}(x, y) &= 0
\end{align*}
\]

3.2. Find an equation of the tangent plane \( h \) to the graph of \( f \) at the point \((a, b, f(a, b))\), where \( a, b \) are some parameters.

Answer:

\[
\begin{align*}
  z &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\
  z &= 3b^2 - 2a^2 + a + (-4a + 1)(x - a) + 6b(y - b)
\end{align*}
\]
3.3. Find parametric equations for the line $\ell$ that goes through the point $P = (a, b, f(a, b))$ and the point $Q = (1, 1, 1)$.

**Answer:**

$$\ell(t) = (1, 1, 1) + t(1 - a, 1 - b, 1 - f(a, b))$$

3.4. For what pairs $(a, b)$ does the line $\ell$ lie in the plane $h$?
Find an equation that describes all such pairs (this equation should involve only $a$ and $b$) and describe/sketch the curve in the plane determined by this equation.

**Answer:**

*One possible solution:*

*Since $\ell$ intersects $h$ (at the point $(a, b, f(a, b))$ it suffices to require that it will also be parallel to $h$.*

*A normal vector to $h$ is* $\langle -4a + 1, 6b, -1 \rangle$

*and the direction of $\ell$ is* $\langle 1 - a, 1 - b, 1 - f(a, b) \rangle$.

*So we want*

$$\langle -4a + 1, 6b, -1 \rangle \cdot \langle 1 - a, 1 - b, 1 - f(a, b) \rangle = 0.$$

*This gives*

$$2a^2 - 4a - 3b^2 + 6b = 0$$

*which can be rewritten as*

$$2(a - 1)^2 - 3(b - 1)^2 = -1$$

*or*

$$\frac{(b - 1)^2}{1/3} - \frac{(a - 1)^2}{1/2} = 1$$

*which is a hyperbola.*
Question 4 (20 pts.)
Consider an isosceles triangle with edge lengths $a$, $a$, and $c$, and with an angle $\theta$, as shown in the figure.

![Isosceles Triangle Diagram]

The law of cosines, applied to such isosceles triangles, states

$$c^2 = 2a^2(1 - \cos \theta).$$

4.1. In view of this equation, we may regard $\theta$ as a function $\theta = \theta(a,c)$ of $a$ and $c$. What is $\theta(1, \sqrt{2})$?

**Answer:**

$$\theta(1, \sqrt{2}) = \frac{\pi}{2}$$

4.2. Using implicit differentiation, find $\frac{\partial \theta}{\partial a}$ and $\frac{\partial \theta}{\partial c}$.

**Answer:**

To find $\frac{\partial \theta}{\partial a}$:

$$\frac{\partial}{\partial a} \left( c^2 \right) = \frac{\partial}{\partial a} \left( 2a^2(1 - \cos \theta) \right) \Rightarrow 0 = 4a(1 - \cos \theta) + 2a^2 \cdot \sin \theta \cdot \frac{\partial \theta}{\partial a}$$

so

$$\frac{\partial \theta}{\partial a} = \frac{-2(1 - \cos \theta)}{a \sin \theta}.$$

To find $\frac{\partial \theta}{\partial c}$:

$$\frac{\partial}{\partial c} \left( c^2 \right) = \frac{\partial}{\partial c} \left( 2a^2(1 - \cos \theta) \right) \Rightarrow 2c = 2a^2 \cdot \sin \theta \cdot \frac{\partial \theta}{\partial c}$$

so

$$\frac{\partial \theta}{\partial c} = \frac{c}{a^2 \sin \theta}.$$
4.3. Now regard $c$ as a function $c = c(a, \theta)$ of $a$ and $\theta$. What is $c(1, \frac{\pi}{3})$? 
(Recall that $\cos(\frac{\pi}{3}) = \frac{1}{2}$.)

**Answer:**

$$c(1, \frac{\pi}{3}) = 1$$

4.4. Find linear approximation of $c = c(a, \theta)$ at the point $(1, \frac{\pi}{3})$.

**Answer:**

$$L(a, \theta) = c(a, \theta) + \frac{\partial c}{\partial a}(a, \theta) \cdot (a - 1) + \frac{\partial c}{\partial \theta}(a, \theta) \cdot (\theta - \frac{\pi}{3}).$$

To compute $\frac{\partial c}{\partial a}$ and $\frac{\partial c}{\partial \theta}$ we use implicit differentiation, this time regarding $a, \theta$ as independent variables and $c$ as a function $c(a, \theta)$.

To find $\frac{\partial c}{\partial a}$:

$$\frac{\partial}{\partial a} \left( c^2 \right) = \frac{\partial}{\partial a} \left( 2a^2 (1 - \cos \theta) \right)$$
$$2c \cdot \frac{\partial c}{\partial a} = 4a(1 - \cos \theta)$$
$$\frac{\partial c}{\partial a} = \frac{2a(1 - \cos \theta)}{c}$$

and thus

$$\frac{\partial c}{\partial a}(1, \frac{\pi}{3}) = 1.$$

To find $\frac{\partial c}{\partial \theta}$:

$$\frac{\partial}{\partial \theta} \left( c^2 \right) = \frac{\partial}{\partial \theta} \left( 2a^2 (1 - \cos \theta) \right)$$
$$2c \cdot \frac{\partial c}{\partial \theta} = 2a^2 \sin \theta$$
$$\frac{\partial c}{\partial \theta} = \frac{a^2 \cdot \sin \theta}{c}$$

and thus

$$\frac{\partial c}{\partial \theta}(1, \frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}.$$ 

Hence

$$L(a, \theta) = 1 + 1 \cdot (a - 1) + \frac{\sqrt{3}}{2}(\theta - \frac{\pi}{3}).$$