Review 1

Problem 1 Parametrization: $x(t) = t$, $y(t) = \frac{2}{3}t^{3/2}$ for $t \geq 0$

$$\vec{r}(t) = \left(t, \frac{2}{3}t^{3/2}\right)$$

$$\vec{r}'(t) = (1, t^{1/2}) \qquad |r'(t)| = \sqrt{1+t}$$

$$s(t) = \int_0^t |r'(u)|\, du = \frac{2}{3}\left[(1+u)^{3/2}\right]_0^t$$

$$= \frac{2}{3}(1+t)^{3/2} - \frac{2}{3}$$

$$\Longrightarrow \left(s + \frac{2}{3}\right)\frac{3}{2} = (1+t)^{3/2}$$

$$1+t = \left(\frac{3}{2}s + 1\right)^{2/3}$$

$$t = \left(\frac{3}{2}s + 1\right)^{2/3} - 1$$

Reparameterization:

$$\left(\left(\frac{3}{2}s+1\right)^{2/3} - 1,\ \frac{2}{3}\left(\left(\frac{3}{2}s+1\right)^{2/3} - 1\right)^{3/2}\right)$$

Problem 2:

Recall $\kappa = \dfrac{|dT/dt|}{|dr/dt|}$ $\qquad N = \dfrac{dT/dt}{|dT/dt|}$

We know $0 = \vec{N} \cdot \vec{T}$

Differentiate: $0 = \dfrac{d\vec{N}}{dt} \cdot \vec{T} + \vec{N} \cdot \dfrac{d\vec{T}}{dt}$

②

so $\frac{d\vec{N}}{dt} \cdot \vec{T} = -\vec{N} \cdot \frac{d\vec{T}}{dt}$

$$= -\vec{N} \cdot \left(\left| \frac{d\vec{T}}{dt} \right| \vec{N} \right)$$

$$= -\left| \frac{d\vec{T}}{dt} \right| \underbrace{\vec{N} \cdot \vec{N}}_{=1}$$

$$= -\left| \frac{d\vec{T}}{dt} \right| = -\kappa(t) \left| \vec{r}'(t) \right|$$

Problem 3: $x(t) = \cos^3 t \quad y(t) = \sin^3 t$

$$\vec{r}(t) = (\cos^3 t, \sin^3 t)$$

$$r'(t) = (-3\sin^2 t \cos t, \ 3\cos^2 t \sin t)$$

$$|r'(t)| = 3\sqrt{\sin^2 t \cos^2 t} = 3|\sin t \cos t|$$

⚠ DON'T FORGET THE ABSOLUTE VALUE!

In fact when $0 \leq t \leq \pi$, we have $\sin t \geq 0$ so

$$s'(t) = |r'(t)| = 3\sin t \, |\cos t|$$

and $L = \int_0^\pi s'(t)\,dt = \int_0^\pi 3\sin t \, |\cos t| \, dt$,

$$= \int_0^{\pi/2} 3\sin t \cos t \, dt - \int_{\pi/2}^{\pi} 3\sin t \cos t \, dt$$

$$= \int_0^{\pi/2} \frac{3}{2}\sin 2t \, dt - \int_{\pi/2}^{\pi} \frac{3}{2}\sin 2t \, dt$$

$$= -\frac{3}{4}\left[\cos 2t\right]_0^{\pi/2} + \frac{3}{4}\left[\cos 2t\right]_{\pi/2}^{\pi}$$

$$= -\frac{3}{4}(-1-1) + \frac{3}{4}(1+1) = 3$$

Problem 4 (1) The curve is an ellipse (in the plane with equation $x+z=0$) so we will find 2 points where the curvature is max and 2 points where it is min.

(2) $\vec{r}(t) = (3\cos t,\ 3\sin t,\ -3\cos t)$ $0 \le t \le 2\pi$

(3) $\overrightarrow{r'(t)} = (-3\sin t,\ 3\cos t,\ 3\sin t)$

$|r'(t)| = 3\sqrt{1+\sin^2 t}$.

$\overrightarrow{r''(t)} = (-3\cos t,\ -3\sin t,\ 3\cos t)$

$$\overrightarrow{r'(t)} \times \overrightarrow{r''(t)} = \begin{pmatrix} -3\sin t \\ 3\cos t \\ 3\sin t \end{pmatrix} \times \begin{pmatrix} -3\cos t \\ -3\sin t \\ 3\cos t \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 0 \\ 9 \end{pmatrix}$$

$$\rightarrow K(t) = \frac{9\sqrt{2}}{3^3\sqrt{1+\sin^2 t}^{\,3}} = \frac{\sqrt{2}}{3\sqrt{1+\sin^2 t}^{\,3}}$$

$K(t)$ is max when $\sin t = 0$ that is to say $t=0$ and $t=\pi$

Points: $(3, 0, -3)$
$(-3, 0, 3)$

(4) we have $\vec{i} = \dfrac{\vec{u} + \vec{w}}{\sqrt{2}}$ $\quad \vec{j} = \vec{v}$

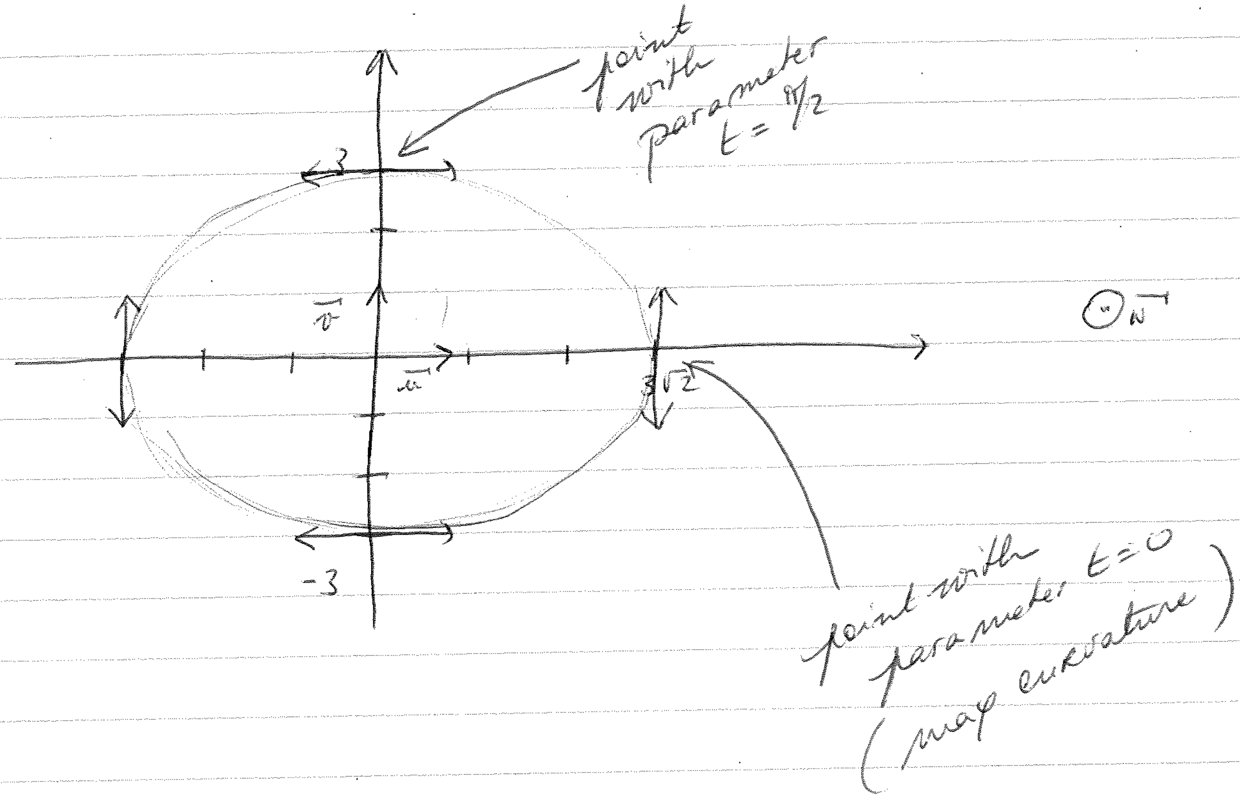
$\vec{k} = \dfrac{\vec{w} - \vec{u}}{\sqrt{2}}$

So $\vec{r}(t) = 3\cos t \left(\dfrac{\vec{u} + \vec{w}}{\sqrt{2}} \right) + 3\sin t\, \vec{v} + 3\cos t \left(\dfrac{\vec{u} - \vec{w}}{\sqrt{2}} \right)$

$= 3\sqrt{2} \cos t\, \vec{u} + 3\sin t\, \vec{v} + 0\, \vec{w}$

Not surprising: the ellipse is contained in the plane $(O, \vec{u}, \vec{v})$ with equation $x + z = 0$

(5)

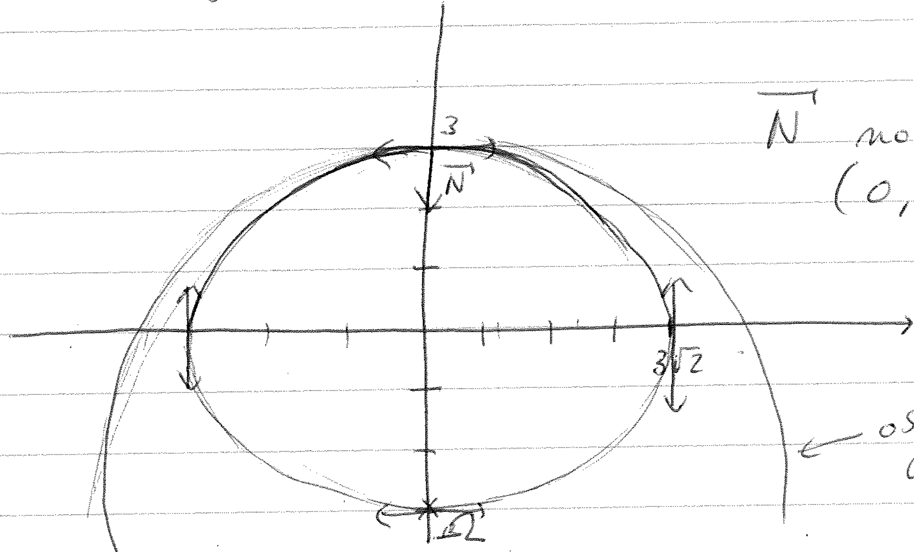


Problem 5: It is the ellipse of problem 4!

Parameterization in the plane (O, x, y):

$x(t) = 3\sqrt{2} \cos t$

$y(t) = 3 \sin t$.



$\overline{N}$ normal vector at $(0,3)$: coordinates $(0,-1)$

← osculating circle at $(0,3)$

Curvature: $\overline{r(t)} = (3\sqrt{2}\cos t, \; 3\sin t)$

$\overline{r'(t)} = (-3\sqrt{2}\sin t, \; 3\cos t)$

$$|r'(t)| = 3\sqrt{1+\sin^2 t}$$

To compute the curvature, we may consider that it is a curve in 3D with z coordinate equal to 0.

then $\overline{r}(t) = (3\sqrt{2}\cos t, \; 3\sin t, \; 0)$ and we apply the formula $\kappa(t) = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$$r'(t) \times r''(t) = \begin{pmatrix} -3\sqrt{2}\sin t \\ 3\cos t \\ 0 \end{pmatrix} \times \begin{pmatrix} -3\sqrt{2}\cos t \\ -3\sin t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 9\sqrt{2} \end{pmatrix}$$

so $$\kappa(t) = \frac{9\sqrt{2}}{3^3\sqrt{1+\sin^2 t}^{\,3}} = \frac{\sqrt{2}}{3\sqrt{1+\sin^2 t}^{\,3}}$$

At point $(0, 3)$ we have $t = \pi/2$

$$K(\pi/2) = \frac{\sqrt{2}}{3\sqrt{2}^{\,3}} = \frac{1}{6}.$$

So the osculating circle has radius 6.

Its center is $\Omega = (0,3) + 6(0,-1) = (0,-3)$

Its equation $x^2 + (y+3)^2 = 36$.

Problem 6: (1)

(a) $$\int_{t=0}^{1} 11t^7 \times 44t^3 + 3t^6 \times 3t^2 \, dt$$

$$= \int_0^1 44 \times 11\, t^{10} + 9t^8 \, dt = 44 + 1 = 45$$

(b) $\vec{r}(t) = (11t, t)$

$$\int_{t=0}^{1} 11t^2 \times 11 + 3t^2 \, dt = \frac{121}{3} + 1 = \frac{124}{3}$$

(2) These two curves have same origin and endpoints. Yet the line integrals of $\vec{F}$ along them is different.

So $\vec{F}$ is not conservative.

Problem 3

$\vec{F} = \left(\cos x + (a-3)y,\ 2+\cos y,\ e^z \right)$

(1) If $\vec{F}$ is conservative then there is $f: \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = \nabla f$.

$$\frac{\partial f}{\partial x} = \cos x + (a-3)y \qquad (i)$$

$$\frac{\partial f}{\partial y} = 2 + \cos y \qquad (ii)$$

$$\frac{\partial f}{\partial z} = e^z \qquad (iii)$$

(iii) implies that there is a function $\varphi: \mathbb{R}^2 \to \mathbb{R}$ s.t

$$f(x,y,z) = e^z + \varphi(x,y)$$

By (ii) $\frac{\partial \varphi}{\partial y} = 2 + \cos y$ so there is $\psi: \mathbb{R} \to \mathbb{R}$ such that $\varphi(x,y) = 2y + \sin y + \psi(x)$

and $f(x,y,z) = e^z + 2y + \sin y + \psi(x)$

But then $\frac{\partial f}{\partial x} = \psi'(x)$ has to be equal to $\cos x + (a-3)y$.

Therefore $a = 3$ and in this case we may pick $\psi(x) = \sin x$

So $\overline{F}$ is conservative if and only if $a=3$ in which case $\overline{F} = \nabla f$

where $f(x,y,z) = e^z + 2y + \sin y + \sin x$

(2) By the fundamental theorem for vector fields:

(a) $$\int_C \overline{F}\cdot \overline{dr} = f(\pi,\pi,2) - f(\pi/2, \pi/2, 0)$$
$$= e^2 + 2\pi - (1+\pi+1+1)$$
$$= e^2 + \pi - 3$$

(b) $\int_C \overline{F}\cdot\overline{dr} = 0$ since it is a closed curve

(c) $$\int_C \overline{F}\cdot\overline{dr} = f(3\pi, -1, 0) - f(0,1,0)$$
$$= 1 - 2 + \sin(-1) - (1 + 2 + \sin(1))$$
$$= -1 - \sin 1 - 3 - \sin 1$$
$$= -4 - 2\sin(1)$$

(3) This is the integral of the vector field
$$\overline{G} = (\cos(x) + 7y,\ 2 + \cos y,\ e^z)$$
$$= \underset{\substack{\uparrow \\ \text{with parameter} \\ a=3}}{\overline{F}} + \underbrace{(7y, 0, 0)}_{\overline{H}}$$

Since $\overline{F}$ is conservative and the curve is closed

$$\int_C \vec{G} \cdot \vec{dr} = \int_C \vec{H} \cdot \vec{dr} = \int_{t=0}^{2\pi} 7(\sin t)(-\sin t)\, dt$$

$$= -7 \int_0^{2\pi} \sin^2 t \, dt$$

$$= -7 \int_0^{2\pi} \left(\frac{1 - \cos 2t}{2} \right) dt$$

$$= -7 \int_0^{2\pi} \frac{1}{2} \, dt = -7\pi$$