Suggested problems or various things to think about

**Problem 1.** Consider the symmetric group $S_3$ and let $k$ be an arbitrary field of characteristic different from 2.

1. What are the 1-dimensional representations of $S_3$? (Show that there is only the trivial representation and the character corresponding to the signature that we will call the sign character).

2. Consider the natural 3-dimensional representation $V_3$ of $S_3$ over $k$. What is the multiplicity of the trivial character of $S_3$ in $V_3$? Does $V_3$ contain the sign character as a subrepresentation? Show that $V_3$ contains a subrepresentation of dimension 2.

   a) If $k$ has characteristic different from 3, show that $V_3$ is semisimple and give its decomposition into irreducibles.

   b) If $k$ has characteristic 3, give a filtration of $V_3$ as a representation of $S_3$ of the form $0 \subset V_1 \subset V_2 \subset V_3$ where $V_i$ has dimension $i$. Describe the quotient representations $V_{i+1}/V_i$. Show that $V_2$ is indecomposable and that $V_3$ is indecomposable.

What happens over a field of characteristic 2?

**Problem 2.** Let $G$ be a finite group. Consider its trivial representation $k_{\text{triv}}$: it is one dimensional and it corresponds to the morphism of groups $G \to k^\times$, $g \mapsto 1$. It is also often denoted by $1_G$. We suppose that the cardinality of $G$ is invertible in the field $k$. Consider the element

$$e_1 := \frac{1}{|G|} \sum_{g \in G} g \in k[G].$$

1. Show that it is a central idempotent of $k[G]$ that is to say that $e_1^2 = e_1$ and that $e_1$ commutes with any element of $k[G]$. Deduce that the trivial representation is a direct summand of the regular representation of $G$ over $k$.

2. Consider an exact sequence of $k$-representations of $G$:

$$0 \to V \to W \to k_{\text{triv}} \to 0$$

This means that there are morphisms of representations $\iota : V \to W$ and $p : W \to k_{\text{triv}}$ such that $\iota$ is injective, $p$ is surjective, and $\text{Im}(\iota) = \text{Ker}(p)$. Show that the exact sequence splits that is to say that there is $s : k_{\text{triv}} \to W$ a morphism of representations such that $\iota \circ s = \text{id}_{k_{\text{triv}}}$. *(This means that $k_{\text{triv}}$ is a projective representation of $G$).* Check that it implies that $W \simeq V \oplus k_{\text{triv}}$ as representations of $G$.

3. Show that if $k_{\text{triv}}$ is a quotient of a $k$-representation $W$, then $W$ contains a copy of $k_{\text{triv}}$ as a direct summand. *This also means that $k_{\text{triv}}$ is a projective representation of $G$.*

To be continued.

**Problem 3.** Let $U$ be the subgroup of upper triangular matrices in $\text{GL}_n(F_p)$. What is the cardinality of $U$? Consider the natural representation of $U$ on the vector space $F_p^n$. Is it decomposable? Is the trivial representation of $U$ over $F_p$ projective?

**Problem 4.** Let $G$ be a group and $k$ a field.
Problem 5. Let $G$ be a finite group and $H$ a subgroup of $G$. Let $k$ be a field. Consider
the induced representation
\[ \text{ind}^G_H(\text{triv}_H) \cong k[G/H] \]
where $\text{triv}_H$ denotes the trivial representation of $H$. A basis for $k[G/H]$ is given by the
characteristic functions $1_{gH}$ where $g$ ranges over a system of representatives of the left
cosets $G/H$.

1. Show that the algebra of endomorphisms $\mathcal{H} := \text{End}_G(k[G/H])$ identifies naturally
with the algebra $k[H\setminus G/H]$ of all functions on $H\setminus G/H$ with values in $k$ (which
can also be seen as functions on $G$ that are constant of the double cosets modulo $H$). The product
of $\varphi$ and $\psi$ in $k[H\setminus G/H]$ is given by
\[ \varphi \ast \psi(x) = \sum_{g \in G/H} \psi(g)\varphi(g^{-1}x). \]

2. Show that there is a natural functor
\[ \mathcal{F} : \text{Rep}_k(G) \rightarrow \text{Mod}(\mathcal{H}), \quad V \mapsto V^H \]
where $\text{Mod}(\mathcal{H})$ is the category of right $\mathcal{H}$-modules.

3. Give a condition under which you can prove that $\mathcal{F}$ is exact. Show that it is not
exact in general.

4. Show that the functor $- \otimes_k k[G/H]$ is left adjoint to $\mathcal{F}$.

5. Let $\text{Rep}_k^H(G)$ denote the full subcategory of $\text{Rep}_k(G)$ of the representations
generated by their $H$-fixed vectors. Show that the restriction of $\mathcal{F}$ to $\text{Rep}_k^H(G)$ is
faithful.

6. Suppose that $G = \text{GL}_2(\mathbb{F}_q)$ and $H = B$ is the upper Borel subgroup.
   (a) Show that $\mathcal{H}$ is two dimensional as a vector space with basis $1_B$ and $1_{B+B}$.
   (b) Compute $1_{B+B} \ast 1_{B+B}$.
   (c) Show that $\mathcal{H}$ is semisimple and describe its simple modules.
(d) Show that the functor above matches up irreducible representations in $\text{Rep}_K^B(G)$ and simple modules of $\mathcal{H}$.

**Problem 6.** Let $G = \text{GL}_n(F_q)$ and $B$ be the upper Borel subgroup with Levi decomposition $B = TU$. Let $\overline{U}$ denote the lower unipotent subgroup.

(1) Consider the symmetric group $\mathfrak{S}_n$ and let $S$ denote the set of all transpositions of the form $s_i := (i, i + 1)$ for $i = 1, \ldots, n - 1$. We say that $(\mathfrak{S}_n, S)$ is a Coxeter system. In particular, it means that any element of $\mathfrak{S}_n$ can be written as a product of elements in $S$. The length of a word $s_{i_1} \ldots s_{i_m}$ for $m \geq 0$ and $1 \leq i_1, \ldots, i_n \leq n - 1$, is the integer $m$. For $w \in \mathfrak{S}_n$, if $w = s_{i_1} \ldots s_{i_m}$ we say that it is a decomposition of length $m$ of $w$. We denote by $\ell(w)$ the minimal length of a decomposition of $w$. A decomposition of $w$ of length $\ell(w)$ is called reduced. The length of $1 \in \mathfrak{S}_n$ is by definition zero.

(a) For $w, w' \in \mathfrak{S}_n$ show that $\ell(ww') \leq \ell(w) + \ell(w')$. Give examples where the inequality is strict (resp. where it is an equality).

(b) If $w = s_{i_1} \ldots s_{i_\ell(w)}$ is a reduced decomposition for $w$ and $1 \leq j < \ell(w)$, show that $s_{i_1} \ldots s_{i_j}$ has length $j$ and $s_{j+1} \ldots s_{i_\ell(w)}$ has length $\ell(w) - j$.

(c) For $s \in S$ and $w \in \mathfrak{S}_n$, show that $\ell(w) - 1 \leq \ell(ws) \leq \ell(w) + 1$.

(2) We consider the set of pairs of integers

$$
\Phi := \{(i, j), i \neq j, 1 \leq i, j \leq n\}.
$$

It is the union of $\Phi^+$ and $\Phi^-$ where

$$
\Phi^+ = \{(i, j) \in \Phi, i < j\} \text{ and } \Phi^- = \{(i, j) \in \Phi, i > j\}.
$$

The set $\Phi$ is the set of roots of $G$ and $\Phi^+$ (resp. $\Phi^-$) is the set of the positive (resp. negative) roots.

To $\alpha = (i, j) \in \Phi$ we attach the subgroup $U_\alpha := 1 + F_q e_{i, j}$ of $G$ where $e_{i, j}$ is the $n \times n$ matrix with zero coefficients except for the coefficient $(i, j)$ which is equal to $1$.

(a) What is the cardinality of $\Phi$, of $\Phi^+$?

(b) Show that to $s \in S$, one can attach naturally an element $\alpha_s$ in $\Phi^+$. We denote by $\Pi$ the set of all $\{\alpha_s, s \in S\}$. Check that every element of $\Phi^+$ is a sum of distinct elements of $\Pi$. Is any sum of distinct elements in $\Pi$ an element in $\Phi^+$?

(c) Check that $U$ is the product of all $U_\alpha$ for $\alpha \in \Phi^+$ (this is the decomposition of $U$ into root subgroups).

(d) Check that there is a natural action of $\mathfrak{S}_n$ on $\Phi$ which is compatible with the action by conjugation of $\mathfrak{S}_n$ on the subgroups $U_\alpha$ for $\alpha \in \Phi$.

(e) Let $s \in S$. What is the set $\{\alpha \in \Phi^+, \ s.\alpha \in \Phi^-\}$?

(f) Check on some examples (e.g. when $n = 2$ and $n = 3$) that for $w \in \mathfrak{S}_n$, we have

$$
\ell(w) = |\{\alpha \in \Phi^+, \ w.\alpha \in \Phi^-\}| = |\{\alpha \in \Phi^+, \ wU_\alpha w^{-1} \subset \overline{U}\}|.
$$

(g) Show that for any $s \in S$ and $w \in \mathfrak{S}_n$, we have $\ell(ws) = \begin{cases} 
\ell(w) + 1 & \text{if } w.\alpha_s \in \Phi^+ \\
\ell(w) - 1 & \text{if } w.\alpha_s \in \Phi^- 
\end{cases}$

(3) Show that for $s \in S$ we have $BsB = \bigcup_{w \in U_\alpha} usB$.

(4) Show that for any $s \in S$ and $w \in \mathfrak{S}_n$ such that $\ell(ws) = \ell(w) + \ell(s)$ we have

$$
BwBsB = BsBwB.
$$
(5) Show that for any $w, w' \in \mathfrak{S}_n$ such that $\ell(ww') = \ell(w) + \ell(w')$ we have
$$BwBw'B = Bww'B.$$ 

(6) What is $BsBsB$?

(7) Let $k$ be an arbitrary field. Show that the Hecke algebra $\mathcal{H}$ of $G$ with respect to $B$
has $k$-basis the set of all characteristic functions $\tau_w := 1_{BwB}$ for $w \in \mathfrak{S}_n$ subject
the relations
$$\tau_w \star \tau_{w'} = \tau_{ww'} \quad \text{if} \ \ell(ww') = \ell(w) + \ell(w'),$$
$$\tau_s^2 = \tau_s \star \tau_s = (q - 1)\tau_s + q \quad \text{for any} \ s \in S.$$ 

(a) What are the one dimensional modules of $\mathcal{H}$?

(b) Suppose that $k$ has characteristic $p$, show that any simple $\mathcal{H}$ module is one
dimensional. Is $\mathcal{H}$ semisimple? Justify your answer.

(c) Suppose that $n = 3$. Give the decomposition of $\mathcal{H}$ into PIMs and identify the
projective covers of the simple modules.

Problem 7. Let $A$ be an abelian group. Decompose the unit of $\mathbb{C}[A]$ as a sum of primitive
orthogonal idempotents.

Problem 8. Let $A$ be a $k$-algebra. A left $A$-module $M$ is flat if the functor $- \otimes_A M$ from
right $A$-modules to $k$-vectorspaces is exact.

(1) Show that if $M$ is flat then for any right ideal $I$ of $A$, the map $I \otimes_A M \to M$ is
injective. We admit that this condition is sufficient for the flatness of $M$ (but you
can think/lookup the proof).

(2) Show that any direct summand of a flat module is flat.

(3) Show that a projective module is flat.

Problem 9. Let $G$ be a finite group and $k$ a field. Show that $k[G]$ is an injective $k[G]$-
module by proving that $\text{Hom}_G(-, k[G])$ is exact. (Note that we can also deduce this from
the fact that $k[G]$ is a symmetric algebra...)

Problem 10. Let $A, B$ be unitary rings and $B \to A$ a morphism of rings making $A$ a left
and right $A$-module. We consider the induction functor $A \otimes_B -$ from left $B$-modules to
left $A$-modules.

(1) Show that $A \otimes_B -$ preserves the projectivity of the modules. Does it take a pro-
tective resolution of the $B$-module $M$ to a projective resolution of the $A$-module
$A \otimes_B M$?

(2) Suppose $A = \mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{Z}$.

(a) Compute $\text{Ext}_A^i(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ for $i \geq 0$.

(b) Describe a non split extension of $\mathbb{Z}/2\mathbb{Z}$ by itself as an abelian group.

(c) Compute $\text{Ext}_A^i(A \otimes_B \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ for $i \geq 0$. What do you notice when you
compare with (a)?

(3) Suppose that $A$ is a free right $B$-module. Show that $A \otimes_B -$ is exact. What can
you say about $\text{Ext}_A^i(A \otimes_B -, -)$?

(4) Same question when $A$ is a projective right $B$-module.

(5) Same question when $A$ is a flat right $B$-module.

(6) Suppose that $G$ is a group and $H$ a subgroup of $G$. Let $k$ be a field. What can you
say about $\text{Ext}_k^i(k[G] \otimes_k[H] -, -)$?
**Problem 11.** Let $G$ be a finite group and $J$ a subgroup of $G$. Let $k$ be a field. Let $i \geq 0$.

1. Show that for a $k|J|$-module $N$ and a $k[G]$-module $M$, we have
   \[ \text{Ext}^i_{k|J}(M, \text{ind}_G^G(N)) = \text{Ext}^i_{k[J]}(M|J, N). \]
2. Show that
   \[ H^i(J, k) = \text{Ext}^i_{k[G]}(k, \text{ind}_G^G(k)). \]

**Problem 12.** Let $q$ be a power of a prime number $p$ and $G = \text{GL}_n(F_q)$. Let $U$ be the subgroup of all upper unipotent matrices and $B$ the Borel subgroup containing $U$ with Levi decomposition $B = TU$. Let $k$ be a field and $\chi : T \to k^\times$ a morphism of groups. We may consider it a morphism of groups $B \to k^\times$ trivial on $U$. For any $w \in S_n$ let $U_w := U \cap w U w^{-1}$.

1. Using Mackey decomposition, show that
   \[ \text{ind}_B^G(\chi)|U \cong \bigoplus_{w \in S_n} \text{ind}_{U_w}^U(k). \]
2. Show that
   \[ H^1(U, \text{ind}_B^G(\chi)) = \bigoplus_{w \in S_n} \text{Hom}(U_w, k). \]
3. What is $H^1(U, \text{ind}_B^G(\chi))$ is $k$ has characteristic different from $p$?
4. Suppose that $k$ has characteristic $p$ and $n = 2$. What is the dimension of $H^1(U, \text{ind}_B^G(\chi))$ over $k$?
5. Suppose that $q = p$ and $n = 2$. How many isomorphism classes of extensions of $k$ by $\text{ind}_B^G(\chi)$ as a representation of $U$ are there? Describe them explicitly.