Problem 1. Recall that the set of complex numbers \((\mathbb{C}, +, \times)\) endowed with the addition and multiplication of complex numbers is a ring with identity element 1. Consider the sets \(\mathbb{Z}[i] = \{a + ib, a \in \mathbb{Z}, b \in \mathbb{Z}\}\) and \(\mathbb{Q}[i] = \{a + ib, a \in \mathbb{Q}, b \in \mathbb{Q}\}\) where \(i\) is the usual complex number satisfying \(i^2 = -1\).

1. Prove that \((\mathbb{Z}[i], +, \times)\) and \((\mathbb{Q}[i], +, \times)\) are subrings of \((\mathbb{C}, +, \times)\).
2. Prove that the following elements are not invertible in \(\mathbb{Z}[i] : 2, 1 + 2i, 1 + i, 5i\).
3. Give a condition on \(a, b \in \mathbb{Z}\) for \(a + ib\) to be invertible in \(\mathbb{Z}[i]\) (and justify your statement, obviously).
4. What are the invertible elements in \(\mathbb{Q}[i]\)?
5. Prove that \(\mathbb{Q}[i]\) is isomorphic to the quotient ring \(\mathbb{Q}[X]/(X^2 + 1)\).

Problem 2. Problem 7 Section 7.1 on the Center of a Ring.

Problem 3. Show that the subring of \(\mathbb{C}\) defined by \(\mathbb{Z}[\sqrt{5}] = \{a + \sqrt{5}b, a, b \in \mathbb{Z}\}\) has infinitely many invertible elements.

Check for yourself that you know how to prove that it is a ring.
For help with such problems, read Example (5) page 227 in Section 7.1 of the group, and page 229 the Example "Rings of quadratic integers".

Problem 4. (1) Recall the definition of \(A^\times\) where \(A\) is a commutative ring with an identity element.
2. Give the list of the elements in \((\mathbb{Z}/52\mathbb{Z})^\times\).
3. What is the inverse of 3 mod 52?
4. What is the inverse of 21 mod 52?

Problem 5. Let \(m, n \in \mathbb{N}, m, n \geq 1\).

1. Prove that the morphism of rings

\[
f : \mathbb{Z}/mn\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}
\]

\[
x \mod mn \rightarrow (x \mod m, x \mod n)
\]

is well defined.
2. Prove that if \(gcd(m, n) = 1\), then this map is injective and surjective.
3. Prove that the system

\[
\begin{align*}
x &\equiv 6 \mod 52 \\
x &\equiv 8 \mod 57
\end{align*}
\]

has a solution.

Problem 6. What are the ideals of \(\mathbb{Z}/n\mathbb{Z}\)? Are they principal?
Problem 7. Problem 1 Section 7.3.

Problem 8. Problem 26 Section 7.3.