Problem 1.

(1) The maximal ideals of $\mathbb{A}$ are the $\mathfrak{m}_I/(x^2, y^3)$, where $\mathfrak{m}_I$ is a maximal ideal of $\mathbb{C}[x, y]$ containing $(x^2, y^3)$

By the Nullstellensatz, $\mathfrak{m}_I$ is of the form $\mathfrak{m}_I = (x-a, y-b)$ for $a, b \in \mathbb{C}$

and $\mathfrak{m}_I \supset (x^2, y^3)$ means $\exists P, Q \in \mathbb{C}[x, y]$ such that $x^2 - y^3 = (x-a)P + (y-b)Q$ so $a^2 - b^3 = 0$

So the maximal ideals of $\mathbb{A}$ correspond to the points of the curve $x^2 - y^3 = 0$.

(2) We want to show that $(x^2, y^3)$ is a prime ideal of $\mathbb{C}[x, y]$

Since $\mathbb{C}[x, y]$ is a UFD, it is equivalent to $x^2 - y^3$ being irreducible in $\mathbb{C}[x, y]$.

Let $\mathfrak{p} := \mathbb{C}[y], x^2 - y^3 \in \mathfrak{p}[x]$ is prime in $\mathfrak{p}$.

To show that it is irreducible in $\mathfrak{p}[x]$, it is therefore enough to show that it is irreducible in $\mathbb{C}(y)[x]$.

Thus $\mathbb{C}(y)[x] = \mathbb{C}(y)[x]$, $x^2 - y^3 \in K[x]$ has degree 2. $K$ (field)

By, it was not irreducible, it would have a root in $K$. Such a root is

a fraction of the form

\[ P(Y) \in K, \quad P, Q \in \mathbb{C}[y], \quad P \neq 0, Q = 1. \]

and we have

\[ P^2 - y^3 Q^2 = 0 \quad \text{in} \quad \mathbb{C}[y] \]

even degree \quad odd degree. \text{ Contradiction.}
3) Let \( \varphi : \mathbb{C}[X,Y] \rightarrow \mathbb{C}[T] \)
\[ \varphi(X, Y) = \mathbb{C}[T^3, T^2] \]

it is a well defined morphism of rings
and its kernel contains \( X^2 - Y^3 \) so it factors through a morphism of rings

\[ \overline{\varphi} : \mathbb{C}[X, Y] \rightarrow \mathbb{C}[T^3, T^2] \]

\[ \text{Im } \overline{\varphi} = \text{Im } \varphi = \mathbb{C}[T^3, T^2] = \text{Subring of } \mathbb{C}[T] \text{ generated by } T^2 \text{ and } T^3 \]

To show that \( \overline{\varphi} \) is injective we need to show that \( \ker \overline{\varphi} = (X^2 - Y^3) \)

Let \( P \in \mathbb{C}[X,Y] \) such that \( \varphi(P) = 0 \).

The Euclidean division of \( P \) by \( X^2 - Y^3 \)
in \( \mathbb{C}[X] \) where \( \mathbb{C}[X] = \mathbb{C}[Y] \) is

\[ P = Q(X^2 - Y^3) + R \]

where \( Q \in \mathbb{C}[X] \), \( R \in \mathbb{C}[X] \) (\( X^2 - Y^3 \) is monic)

and \( \deg R \leq 1 \).

\[ \deg X \]

so \( R = U(Y) X + V(Y) \) where \( U, V \in \mathbb{C}[Y] \)

We have \( R(T^3, T^2) = U(T^2)T^3 + V(T^2) = 0 \) in \( \mathbb{C}[T] \)

But \( U(T^2)T^3 \) has odd degree
\( V(T^2) \) has even degree

so \( U = V = 0 \) and \( P = 0 \)

and \( P \in (X^2 - Y^3) \)
4) The fraction field $K$ of $A$ is the smallest field in which the non-zero elements of $A$ are invertible.

Therefore $K \subseteq C(T)$.

Now $T = \frac{T^3}{T^2}$ so $T \in K$ and therefore any $P(T) \in K$ for $P \in C[T]$ and any fraction $\frac{P(T)}{Q(T)}$ where $P \in C[T]$, $Q \in C[T]$ and $Q \neq 0$ also lies in $K$.

So $K = C(T)$.

By (3) $A = C[T^3, T^2]$.

$T^3$ is irreducible in $C[T^3, T^2]$.

($T^3 = P(T^3, T^2)Q(T^3, T^2)$ --- look at the degrees)

Yet $C[T^3, T^2]/(T^3)$ is not an integral domain since $T^2 \times T^3 \subseteq (T^3)$ yet $T^2 \notin (T^3)$.
Problem 2

(1) \( k \to R \rightrightarrows \frac{k}{R} \) is a morphism of rings.

So a \( R \)-module is a \( k \)-vector space.

Let \( M = k^v \) a 1-dimensional vector space which is also a \( k[G] \)-module.

For \( g \in G \) seen in \( k[G] \) acts on \( v \) and \( g \cdot v \in k^v \) so there is \( \lambda(g) \in k \) such that \( g \cdot v = \lambda(g)v \).

But for \( g' \in G \), \( g' \cdot (g \cdot v) = g' \cdot (\lambda(g)v) = \lambda(g)g' \cdot v = \lambda(g)g' \cdot v \)

\( (g'g) \cdot v = \lambda(g') \lambda(g) \cdot v \)

So \( \lambda : G \to k^* \) is a morphism of groups.

(Note that \( \lambda(g) \neq 0 \) since \( \lambda(gg') = \lambda(g) \lambda(g') = 1 \) since the identity \( e \) of \( G = k[G] \) acts by \( (1, m) 1 \mapsto m \).

\( G = S_3 \) is generated by \((1,2)\) and \((1,2,3)\)

( Classic result we approve it by hand: \( S_3 = \{(1,2), (2,3), (1,3) = (1,2)(2,3)(1,2) \)
\((1,2)(2,3) = (1,2,3), (2,3)(1,2) = (1,3,2)\)
\( \text{id} = (1,2)^2 \) )
Furthermore, 
\[ (1,2,3) (12) (1,2,3)^{-1} \]
\[ = (1,2,3) (12) (13,2) \]
\[ = (2,3) \]

So, a morphism of groups is determined by the images of \((1,2)\) and \((1,2,3)\) and in fact 
\[ \lambda((1,2,3)) = \lambda((123)) \lambda((1,2)) \lambda((123))^{-1} \]
\[ = \lambda((1,2)) \]

So \(\lambda : G \to \mathbb{Z}^*\) is determined by \(\lambda((1,2))\)

Since \((1,2)^2 = 1\) we have either 
\[ \lambda((1,2)) = 1 \] and then \(\lambda : G \to \mathbb{Z}^*\)
\[ g \mapsto 1 \]

or 
\[ \lambda((1,2)) = -1 \] and then \(\lambda : G \to \mathbb{Z}^*\)
\[ g \mapsto (-1)^{\text{sign}(g)} \]

(2)(a) Let \(v = e_1 + d_1 e_2 + d_3 e_3\) and suppose that 
\[ k v \cong \mathbb{Z}^* \] as an \(R\)-module

Then 
\[ (1,2) \cdot v = -v \]
\[ d_2 e_1 + d_1 e_2 + d_3 e_3 \]
\[ (2,3) \cdot v = -v \]
\[ d_1 e_1 + d_3 e_2 + d_2 e_3 \]
\[ \Rightarrow d_2 = -d_1 \]
\[ d_3 = 0 \]
\[ d_1 = 0 \]
\[ \Rightarrow d_1 = d_2 = d_3 = 0 \]

So \(v\) does not contain a copy of \(\mathbb{Z}^*\).
On the other hand, it is easy to see that $k[(a + t)(a + t^2)]$ is a submodule isomorphic to $V_2$ AND that it is the only copy of their contained in $V_3$.

6) Let $V_2 = \text{Span} \left \{ a - e_2, e_2 - e_3 \right \}$ (proved as above).

We need to show that $V_2$ is stable by the action of $R$. It is enough to see that:

\[ (1, 2), V_2 \subseteq V_2 \quad \text{since } (1, 2) \text{ and } (2, 3) \text{ generate } G \]

But

\[ (1, 2), e_1 - e_2 = (a - e_2) \subseteq V_2 \]
\[ (1, 2), e_2 - e_3 = e_1 - e_3 = (a - e_2) + (e_2 - e_3) \subseteq V_2 \]
\[ (2, 3), e_1 - e_2 = e_1 - e_3 \subseteq V_2 \]
\[ (2, 3), e_2 - e_3 = -(e_1 - e_3) \subseteq V_2 \]

(i) Check $k \neq 3$. Let $V_1 = k(a + e_2 + e_3)$.

We show $V_1 = V_1 \oplus V_2$ as a vector space.

It is enough to check that $a + e_2 + e_3 \notin V_2$.

\[ a + e_2 + e_3 = \alpha (a - e_2) + \beta (e_2 - e_3) \quad \alpha, \beta \in k \]

Since $V_1$ and $V_2$ are submodules, this is a decomposition of $V_3$ as a direct sum of modules.

To see that $V_2$ is simple: If it wasn't, it would contain a 1-dimensional submodule but we saw above that $V_1$ is the only 1-dimensional submodule of $V_3$.

So $V_1$ and $V_2$ are simple.
We notice that \( V_2 \leq V_1 \)

\[
V_2/V_1 = \left\{ (a - e_2) \mod V_1 \right\}
\]

\[
(1, 2) \cdot (a - e_2) \mod V_1 = -(a - e_2) \mod V_1
\]

\[
(2, 3) \cdot (a - e_2) \mod V_1 = e_2 - e_3 \mod V_1
\]

\[
e_2 - e_2 = e_2 - e_3 \mod V_1
\]

\[
e_2 - e_2 = (a - e_2) \mod V_1
\]

So, \( V_2/V_1 = k \text{sign} \).

\[
V_3/V_2 = k \left( a \mod V_2 \right)
\]

\[
(1, 2) \cdot e_1 \text{ mod } V_2 = e_2 \text{ mod } V_2 = e_1 \text{ mod } V_2
\]

\[
(2, 3) \cdot e_1 \text{ mod } V_2 = e_3 \text{ mod } V_2 = e_1 \text{ mod } V_2
\]

\[
V_3/V_2 = k \text{thetiv}.
\]

If the sequence

\[
0 \rightarrow V_1 \rightarrow V_3 \rightarrow V_3/V_1 \rightarrow 0
\]

was exact, we would have

\[
V_3 = V_1 \oplus V_3/V_1
\]

But \( V_2/V_1 \leq V_3/V_1 \), so \text{sign} would be contained in \( V_3 \), contradiction.
So the sequence does not split.

Remark about this problem: if \( \text{char } R = 2 \) then \( \text{triv } \cong \ker \).

**Problem 3.**

1) \( \mathbb{N} \ni q \to x \mapsto U_m = \sqrt{2} \pi \frac{q}{\sqrt{2} \pi x} = \sqrt{2} \pi \frac{q}{2^{m/2} x}, \quad x \geq 0 \)

\( \forall x \in \mathbb{R}^+ \) is determined by the image of \( \xi = e^{2 \pi i m} \) which has to lie in \( U_m \).

So \( \chi(\xi) = \xi^N \) for some \( N \in \{0, \ldots, n-1 \} \)

So then \( \chi: U_m \to \mathbb{C}^x \)

\( \chi(x) = x^N \) for \( N \in \{0, \ldots, n-1 \} \).

2) \( \circ \) \( g \cdot \xi = \frac{1}{n} \sum_{g \in G} \chi(g) \cdot g \cdot \xi \)

\( = \frac{1}{n} \sum_{g \in G} \chi(g) \cdot x(g_0) \cdot y^{-1} \)

\( = \chi(g_0) \cdot \xi \)

So \( \text{ker } \xi \) is a 1-dimensional \( \mathbb{C}G \)-module such that \( g_0 \cdot \xi = \chi(g_0) \cdot \xi \).
6) \( e_x e_{x'} = \frac{1}{m} \sum_{g \in G} x(g^{-1}) e_x e_{x'} \)

\( x' : G \rightarrow C^x \)

\( x' : G \rightarrow C^x \) or \( \frac{1}{m} \sum_{g \in G} x(g^{-1}) x'(g) e_x \)

by a)

\[ x = x' \quad e_x e_x = \frac{1}{m} \sum_{g \in G} e_x = e_x \]

so \( e_x \) is an idempotent

3) if \( x \neq x' \)

\( x' x^{-1} : G \rightarrow C^x \)

\( g \mapsto x(g) x'(g) \)

is of the form \( x \mapsto x^n \) for \( n \in \{1, \ldots, m-1\} \)

\[ e_x e_{x'} = \frac{1}{m} \sum_{g \in G} e_x e_{x'} \]

\[ = \frac{1}{m} \left( \sum_{k=0}^{m-1} e_x e_{x'}^{kN} \right) e_x e_{x'} \]

Recall that we identify

\[ G = \mathbb{U}_m = \langle \frac{\xi}{N} \rangle \]

\[ N \equiv 1 \mod m \]

\[ e_{x^n} = \frac{1}{m} \frac{1 - (\frac{\xi}{N})^m}{1 - \frac{\xi}{N}} e_{x'} \]

\[ e_{x^n} = 0 \]

So \( e_x \) and \( e_{x'} \) are orthogonal idempotents.
We claim that
\[ \mathbb{C} [6] = \bigoplus \text{ker } \alpha \text{ as } \mathbb{C} [6]-\text{modules.} \]

Proof: Suppose \( \sum \lambda_x e^x = 0 \) where \( \lambda_x \in \mathbb{C} \)

Let \( \lambda_x : \mathbb{C} \to \mathbb{C} \)
\[ 0 = \lambda_x \left( \sum \lambda_x e^x \right) = \lambda_x \cdot e^x \implies \lambda_x = 0. \]

So the \( \lambda_x \) are \( \mathbb{C} \)-linearly independent vectors in \( \mathbb{C} [6] \).

Since there are \( m \) such \( \lambda_x : \mathbb{C} \to \mathbb{C} \), it is a collection of \( m \) linearly independent vectors in \( \mathbb{C} [6] \). So it is a basis.

\[ \mathbb{C} [6] = \bigoplus \text{ker } \alpha \text{ as } \mathbb{C} \text{-vector spaces.} \]

But we checked that \( \text{ker } \alpha \) is a submodule in 2) a) so it is a decomposition as submodules.

Problem 7 Section 10.3
\[ 0 \to N \overset{i}{\to} M \overset{\pi}{\to} M/N \to 0. \]

Since \( M/N \) is \( \mathbb{F} \)-series \( m_1, \ldots, m_r \in M \)
Suppose that \( M/N = \sum_{i=1}^{r} \mathbb{F} \pi(m_i) \)
Let $n_1, \ldots, n_k \in N$ such that
\[ N = \sum_{i=1}^{t} R_{n_i} \]  
(Since $N$ is $fg$)

We want to show that
\[ M = \sum_{i=1}^{t} R_{n_i} + \sum_{i=1}^{s} R_{m_i}. \]

This is because for $m \in M$ we can find $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that
\[ \overline{n}(m) = \sum_{i=1}^{t} \lambda_i \overline{n}(m_i) \]

\[ \Rightarrow m - \sum_{i=1}^{t} \lambda_i m_i \in \ker \overline{n} = N - \sum_{i=1}^{s} R_{m_i}. \]

So $m \in \sum_{i=1}^{t} R_{n_i} + \sum_{i=1}^{s} R_{m_i}$. \(\square\)