Matrix algebra, Math 221

Worksheet 4

Given a \( m \times n \) matrix \( A \), and the corresponding linear transformation
\[
T_A : \mathbb{R}^n \to \mathbb{R}^m
\]
we recall:

- the Rank of \( A \) is the dimension of the Range of \( T_A \), i.e the dimension of the vector space spanned by the columns of \( A \).
- the Null space of \( A \), also called the Kernel of \( T_A \), is the set of \( x \in \mathbb{R}^n \) such that \( A x = 0 \).

We admit the following formula (which you have to know):

\[
n = \text{Rank}(A) + \dim(\text{Null}(A))
\]

**Problem 1.** Let
\[
A = \begin{pmatrix}
2 & 3 & 1 \\
0 & 2 & 2 \\
1 & 2 & 1
\end{pmatrix}.
\]

(1) Find a basis of the Null space of \( A \).
(2) What is the rank of \( A \)?
(3) Show that the Range of \( T_A \) is the plane with equation \( 2x + y - 4z = 0 \) (this is harder than an exam question).
(4) Find a basis of the Range of \( T_A \).

**Problem 2.** Let \( A \) be a square matrix. Show that \( A \) is invertible \( \iff \text{Null}(A) = \{0\} \iff \text{Rank}(A) = n \)

**Problem 3.**
(1) Show that the matrix \( A = \begin{pmatrix}
1 & 2 & 3 \\
0 & 1 & 2 \\
1 & 2 & 6
\end{pmatrix} \) is invertible.
(2) Compute its determinant.