Problem set 5, functions, equivalence relations + review, not to be handed in.

Problems marked with a (*) should be at your level and can be used to review on Equivalence Relations, Functions, and Cardinality of Sets, as well as Divisibility in \( \mathbb{Z} \).

**Problem 1 (\(*\)).** Prove that for all \( x,y \in \mathbb{R} \) we have \(|x + y| \leq |x| + |y|\) and deduce that \(|x - y| \geq ||x| - |y||\).

**Problem 2.** Let \( X = \{(a, b), \ a \in \mathbb{Z}, \ b \in \mathbb{N} \ \text{with} \ b \geq 1\} \). Define a relation \( \mathcal{R} \) on \( X \) by 
\[
(a, b) \mathcal{R} (a', b') \text{ if } ab' = a'b.
\]

1. (*) Check that \( \mathcal{R} \) is an equivalence relation. Denote by \( X/\mathcal{R} \) the set of equivalence classes.
2. Check that the function
\[
f : \ X/\mathcal{R} \rightarrow \mathbb{Q} \quad x = [(a, b)]_\mathcal{R} \mapsto \frac{a}{b}
\]
is well defined : what you have to do is show that the quantity \( \frac{a}{b} \) does not depend on the choice of the representative you pick in the class \( [(a, b)]_\mathcal{R} \) (namely : take a representative \((a', b') \in X \) of the class \([ (a, b)]_\mathcal{R} \) and check that \( \frac{a}{b} = \frac{a'}{b'} \)).

3. Is \( f \) injective ? surjective ?

**Problem 3 (\(*\)).** We define the sequence \((u_n)_{n \in \mathbb{N}}\) the following way :
\[
\begin{align*}
u_0 &= 2 \\
u_{n+1} &= 1 + \frac{1}{1 + u_n} \quad \text{for} \ n \geq 0
\end{align*}
\]
Prove that for any \( n \in \mathbb{N} \) we have \( 1 \leq u_n \leq 2 \).

**Problem 4 (\(*\)).** For \( p \in \mathbb{N} \) a prime number and \( a \in \mathbb{Z} \), show that 
\[ p \text{ divides } a \text{ or } \gcd(a, p) = 1. \]

**Problem 5 (\(*\)).** Let \( a, b, c \in \mathbb{Z} \) and suppose that \( a \neq 0 \). Show that 
\[ a \text{ divides } bc \text{ and } \gcd(a, b) = 1 \Rightarrow a \text{ divides } c . \]

*Hint : \( \gcd(a, b) = 1 \) implies that there is \((x, y) \in \mathbb{Z}^2 \) such that \( 1 = ax + by \). Now write \( c \) using \( a, b, x, y \) and \( c \).*

**Problem 6 (\(*\)).** Let \( p \in \mathbb{N} \) a prime number and \( a, b \in \mathbb{Z} \). Show that 
\[ p \text{ divides } ab \Rightarrow p \text{ divides } a \text{ or } p \text{ divides } b . \]

*Hint : use the previous problems.*
Problem 7 (*).  
(1) Let \( p \in \mathbb{N} \) a prime number and \( a \in \mathbb{Z} \). Find all the solutions to the equation 
\[
a^2 = x^2 \mod p.
\]
What are these solutions modulo \( p \)? Hint: use the previous problem.

(2) Find all the solutions modulo \( 8 \) to the equation 
\[
1^2 = x^2 \mod 8.
\]
What is different from what happens in the previous question?

Problem 8.  Let \( p \in \mathbb{N} \) a prime number.

(1) Check that the function 
\[
f : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \quad [a]_p \mapsto [a^2]_p
\]
is well defined. Its image is by definition the set of all squares in \( \mathbb{Z}/p\mathbb{Z} \).

(2) Let \( (\mathbb{Z}/p\mathbb{Z})^\times \) denote the set \( \mathbb{Z}/p\mathbb{Z} \setminus \{0\}_p \). Show that \( f \) restricts to a map 
\[
f^\times : (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times.
\]
Hint: use Problem 6.

(3) We want to count the elements in the image of \( f^\times \), namely the number of squares in \( (\mathbb{Z}/p\mathbb{Z})^\times \).

(a) Given an element \( s \) in the range/image of \( f^\times \), what is the cardinality of its preimage \( f^{-1}\{s\} \) by \( f^\times \)?

Hint: use Problem 6.

(b) Deduce the number of squares in \( (\mathbb{Z}/p\mathbb{Z})^\times \).

(c) What is the cardinality of the range/image of \( f \)?

Problem 9 (*).  
(1) Let \( n, x, y, z \in \mathbb{N} \) with \( n, x, y, z \geq 1 \). Suppose that \( x^2 + y^2 = z^n \). Then compute \( (xz)^2 + (yz)^2 \) in terms of \( z \).

(2) Let \( n \geq 1 \). Prove that the equation 
\[
x^2 + y^2 = z^n
\]
has a solution \( (x, y, z) \in \mathbb{Z} \) that satisfies \( x, y, z \geq 2 \). Split into cases, depending on the parity of \( n \) and do two proofs by induction, one for \( n \) even and another one for \( n \) odd (ex: for \( n \) even: prove by induction on \( m \geq 1 \) that the equation 
\[
x^2 + y^2 = z^{2m}
\]
has a solution \( (x, y, z) \in \mathbb{Z} \) that satisfies \( x, y, z \geq 2 \)).

Problem 10 (*).  Define on \( \mathbb{R} \) the relation : \( x \Re y \) if \( \cos^2(x) + \cos^2(y) = 1 \). Is it an equivalence relation?

Problem 11 (*).  On the set of all lines in the \( xy \) plane, we define the relation : \( D \Re D' \) if \( D \) and \( D' \) are orthogonal.

Problem 12 (*).  Let \( f : X \rightarrow Y \) be a function.

(1) Recall what the following statements mean

(a) \( f \) is injective.
(b) \( f \) is surjective.
(c) \( f \) is bijective.

(2) We consider the two following functions
\[
    u : \mathbb{N} \rightarrow \mathbb{N} \\
    n \mapsto \begin{cases} 
    \frac{n}{2} & \text{if } n \text{ is even} \\
    \frac{n-1}{2} & \text{if } n \text{ is odd}
    \end{cases} \quad \text{and} \quad v : \mathbb{N} \rightarrow \mathbb{N} \\
    n \mapsto 2n.
\]

(a) Describe explicitly \( u \circ v \).
(b) What is the range of \( v \circ u \)?
(c) Is \( u \) injective? surjective? bijective?
(d) Same question for \( v \).

Problem 13. Let \( \mathcal{R} \) be the relation defined on \( \mathbb{R} \times \mathbb{R} \) by
\[
(x_1, y_1) \mathcal{R} (x_2, y_2) \quad \text{if} \quad x_1^2 + y_1^2 = x_2^2 + y_2^2.
\]

(1) (*) Check that it is an equivalence relation.
(2) (*) Describe the equivalence classes. You can make a drawing.
(3) We denote by \( (\mathbb{R} \times \mathbb{R})/\mathcal{R} \) the set of all equivalence classes of the previous relation.
Which of the following functions are well-defined? Justify your answer.
\[
    f : (\mathbb{R} \times \mathbb{R})/\mathcal{R} \rightarrow \mathbb{R} \\
    (x, y) \mapsto x^2 + y^2 \\
    g : (\mathbb{R} \times \mathbb{R})/\mathcal{R} \rightarrow \mathbb{R} \\
    (x, y) \mapsto x + y \\
    h : (\mathbb{R} \times \mathbb{R})/\mathcal{R} \rightarrow \mathbb{R} \\
    (x, y) \mapsto x^4 + y^4 + 2x^2y^2 + 7
\]

Problem 14. Recall the following definitions: two sets \( A \) and \( B \) have the same cardinality if there is a bijection \( A \rightarrow B \) (or equivalently a bijection \( B \rightarrow A \) since a bijection always has an inverse function); a set \( A \) is called countable if it has the same cardinality as \( \mathbb{N} \) namely if there is a bijection \( A \rightarrow \mathbb{N} \).

(1) (*) Consider the function
\[
    f : \mathbb{N} \rightarrow \mathbb{Z} \\
    x \mapsto \begin{cases} 
    x/2 & \text{if } x \text{ is even} \\
    (-x-1)/2 & \text{if } x \text{ is odd}
    \end{cases}
\]
Prove carefully that \( \mathbb{Z} \) this is a bijection and deduce that \( \mathbb{Z} \) is countable.

(2) (*) Consider the function
\[
    f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \\
    (x, y) \mapsto 2^x(2y + 1).
\]
Prove that \( \mathbb{N} \times \mathbb{N} \) is countable.
(3) (More difficult) Let $X$ be a set and $\mathcal{P}(X)$ the set of all subsets of $X$. Denote by $\{0,1\}^X$ the set of all functions on $X$ with values in $\{0,1\}$. For $Y \subseteq X$, define $f_Y$ to be the function

$$f_Y : X \to \{0,1\}$$

$$x \mapsto \begin{cases} 
1 & \text{if } x \in Y \\
0 & \text{if } x \in X \setminus Y
\end{cases}$$

(a) Prove that the function

$$F : \mathcal{P}(X) \to \{0,1\}^X$$

$$A \mapsto f_A$$

is a bijection.

(b) If $X$ has finite cardinality equal to $n$, how many elements are there in $\{0,1\}^X$?

(c) If $X$ has finite cardinality equal to $n$, we proved in class by induction that $\mathcal{P}(X)$ has cardinality $2^n$. Give another proof for this result.

**Problem 15** (*). Let $f : X \to Y$ be a function.

(1) **Image of a set by a function**

(a) For $x \in X$, what is $f(\{x\})$?

(b) How is $f(X)$ usually called?

(c) Let $\alpha : \mathbb{R} \to \mathbb{R}$ defined the following way : for $x \in \mathbb{R}$, $\alpha(x)$ is the largest integer $n \in \mathbb{Z}$ such that $x \geq n$.

   (i) Give the image of the following elements : $1$, $-5$, $-2.5$, $\pi$, $-\pi$.

   (ii) Is the function $\alpha$ injective? surjective?

   (iii) Draw the graph of $\alpha$.

   (iv) What is the image under $\alpha$ of the following intervals

        $(0,1)$, $[0,1]$, $(0,1]$, $[0,1)$, $(0, +\infty)$?

(2) **Pre-image of a set by a function**

(a) If $f$ is injective and $y \in Y$, what can you say about $f^{-1}(\{y\})$?

(b) Let $Y' \subseteq Y$. Prove that $Y' \supseteq f(f^{-1}(Y'))$.

   If $f$ is surjective show $f(f^{-1}(Y')) = Y'$. Compare with Problem 6 of Section 12.6.

(c) Let $X' \subseteq X$. Prove that $X' \subseteq f^{-1}(f(X'))$.

   If $f$ is injective show $f^{-1}(f(X')) = X'$. Compare with Problem 5 of Section 12.6.

   *See also other problems of that section, for example Problems 13 and 14.*

(d) With our previous example $\alpha$, give

   $\alpha^{-1}\{0\} = \ldots$

   $\alpha^{-1}\{-2\} = \ldots$

   $\alpha^{-1}\mathbb{N} = \ldots$

   $\alpha^{-1}\mathbb{Z} = \ldots$

   $\alpha^{-1}\{0.5\} = \ldots$
(3) **Restriction of the domain of a function** Given a function $f : X \rightarrow Y$ and $W$ a subset of $X$, the restriction of $f$ to $W$ is denoted by $f|_W$. It is the function 

$$f|_W : W \rightarrow Y \quad w \mapsto f(w).$$

(a) With our function $\alpha$, describe $\alpha|_Z$.

(b) Consider the function

$$\cos : \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto \cos(x)$$

(i) What is the range of $\cos$?

(ii) Draw the graph of $\cos$.

(iii) Give a subset $W \subseteq \mathbb{R}$ such that $\cos|_W$ is injective.

(4) **Inverse of a function** Consider the following functions

$$\cos : \mathbb{R} \rightarrow \mathbb{R} \quad \beta : [0, +\infty) \rightarrow [0, +\infty) \quad \gamma : \mathbb{N} \rightarrow \mathbb{N} \quad \delta : \mathbb{Z} \rightarrow \mathbb{Z}$$

$$x \mapsto \cos(x) \quad x \mapsto x^2 \quad n \mapsto 2n^2 + 1 \quad n \mapsto n + 54$$

$$\epsilon : \{1, 2, 3, 4\} \rightarrow \{a, b, c, d\} \quad \zeta : \{1, 2, 3, 4\} \rightarrow \{a, b, c, d\}$$

$$\begin{array}{c|cccc}
1 & \rightarrow & a & 1 & \rightarrow & a \\
2 & \rightarrow & b & 2 & \rightarrow & b \\
3 & \rightarrow & c & 3 & \rightarrow & c \\
4 & \rightarrow & c & 4 & \rightarrow & d.
\end{array}$$

(a) Which of these functions are bijective?

(b) Give explicitly the following sets:

- $\text{graph}(\epsilon)$ and $\{(y, x), \text{ such that } (x, y) \in \text{graph}(\epsilon)\}$.

Is the second one the graph of a function?

(c) For the functions above that are bijective, describe their inverse functions.

(d) What is $\epsilon^{-1}(\{c\})$?