We recall the following theorem that we proved in class:

**Theorem.** Given \( a \in \mathbb{Z} \) and \( b \in \mathbb{Z} \) with \( b \geq 1 \). There is a unique pair \((q, r) \in \mathbb{Z}^2\) satisfying 
\[ a = bq + r \quad \text{and} \quad 0 \leq r < b. \]

The element \( q \) is called the **quotient** and the element \( r \) is called the **remainder** of the division.

We say that \( b \) divides \( a \) (and we sometimes write \( b | a \)) when the remainder \( r \) is equal to zero. It means that \( a \) is a multiple of \( b \).

**Problem 1.**
1. Write the Euclidean division of 237 by 11.
2. Does 13 divide 348?
3. Let \( a \in \mathbb{Z} \). What are the possible remainders for the division of \( a \) by 3? Give an example of \( a \) for each of the remainders of your list.
4. Suppose that \( a \in \mathbb{Z} \) is odd which means that 2 does not divide \( a \). Let \( a = 2q + r \) the Euclidean division of \( a \) by 2.
   (a) What is \( r \)?
   (b) Using \( q \), write the Euclidean division of \( a^2 \) by 2 (identify the quotient and the remainder clearly).
   (c) Is \( a^2 \) odd or not?

**Problem 2.** Recall that for \( x \in \mathbb{R} \), we define the absolute value \( |x| \) of \( x \) by
\[ |x| = \begin{cases} 
   x & \text{if } x \geq 0 \\
   -x & \text{if } x \leq 0 
\end{cases} . \]

For \( a \in \mathbb{R} \) we define the sets
\[ A_a = \{ x \in \mathbb{R}, \ 0 \leq |x| \leq -a^2 + a + 2 \} \ \text{and} \ B_a = \{ a \} \times A_a. \]

1. What is \( A_1 \)? (write it as an interval of \( \mathbb{R} \))
2. What is \( \mathbb{R} \setminus A_1 \)?
3. What is \( A_{-1} \)? (write it as an interval of \( \mathbb{R} \))
4. What is \( A_2 \)? (write it as an interval of \( \mathbb{R} \))
5. Is \( A_a \) non-empty for all \( a \in \mathbb{R} \)?
6. Draw the curve with equation \( y = -x^2 + x + 2 \).
7. Draw \( \bigcup_{a \in [-1,2]} B_a \).

**Problem 3.** Book Section 1.5 Question 8

**Problem 4.** Book Section 2.5 Question 8

**Problem 5.** Book Section 2.9 Questions 6 and 10

**Problem 6.** Book Section 2.10 Questions 2, 4, 6 and 8
Problem 7 (Due Friday June 1st). For \( a \in \mathbb{N} \), we denote by \( a\mathbb{Z} \) the following set:
\[
a\mathbb{Z} = \{ax : x \in \mathbb{Z}\}.
\]
For \( a \in \mathbb{N} \) and \( b \in \mathbb{N} \), we denote by \( a\mathbb{Z} + b\mathbb{Z} \) the following set:
\[
a\mathbb{Z} + b\mathbb{Z} = \{ax + by : x, y \in \mathbb{Z}\}.
\]
(1) Is \( 2\mathbb{Z} \cup 3\mathbb{Z} \) equal to \( 2\mathbb{Z} + 3\mathbb{Z} \)?
(2) Write the Euclidean division of 25 by 6 and show that \( 1 \in 6\mathbb{Z} + 25\mathbb{Z} \).
(3) Show that \( \mathbb{Z} = 6\mathbb{Z} + 25\mathbb{Z} \).
(4) Is it true that \( \mathbb{Z} = 6\mathbb{Z} + 24\mathbb{Z} \)?

For your convenience, I recall the proof that we did in class:

Proof of the existence of a pair \((q, r)\) satisfying \( a = bq + r \) and \( 0 \leq r < b \). Consider the set
\[
\mathcal{R} = \{a - xb : x \in \mathbb{Z}\}.
\]
- We first check that \( \mathcal{R} \cap \mathbb{N} \) is not empty. We can find \( n \in \mathbb{Z} \) such that the quotient \( a/b \) (which is an element of \( \mathbb{Q} \)) lies in \([n, n + 1)\). (e.g. if \( a/b = 2.123 \) then \( n = 2 \); if \( n = -1.4567 \) then \( n = -2 \).) We then have \( a/b \geq n \), so \( a \geq nb \) (because \( b \geq 1 \) and
\[
a - nb \in \mathcal{R} \cap \mathbb{N},
\]
so \( \mathcal{R} \cap \mathbb{N} \) is not empty.
- By the well-ordering principle, the set \( \mathcal{R} \cap \mathbb{N} \) has a smallest element namely:
\[
\text{there is } r \in \mathcal{R} \cap \mathbb{N} \text{ such that for any } y \in \mathcal{R} \cap \mathbb{N} \text{ we have } y \geq r.
\]
Since this element \( r \) lies in \( \mathcal{R} \) we may write it in the form \( a - xb \) for some \( x \in \mathbb{Z} \) and we are going to call \( q \) this element \( x \) (just because this is the notation used in the theorem).
We then have
\[
a = bq + r
\]
and it remains to show that \( 0 \leq r < b \).
- It is clear that \( r \geq 0 \) since \( r \in \mathbb{N} \).
- Now notice that \( r - b < r \) so \( r - b \) cannot lie in \( \mathbb{N} \cap \mathcal{R} \) (whose elements are all \( \geq r \)).
But \( r - b = a - (q + 1)b \) so \( r - b \) lies in \( \mathcal{R} \). It lies in \( \mathcal{R} \) but not in \( \mathbb{N} \cap \mathcal{R} \) so it means that it does not lie in \( \mathbb{N} \). It is therefore an element of \( \mathbb{Z} \) which is not in \( \mathbb{N} \); we have proved that \( r - b < 0 \) and \( r < b \).

Proof of the uniqueness of the pair \((q, r) \in \mathbb{Z}^2 \) satisfying \( a = bq + r \) and \( 0 \leq r < b \). Let \( (q, r) \in \mathbb{Z}^2 \) satisfying \( a = bq + r \) and \( 0 \leq r < b \) and \((q_0, r_0) \in \mathbb{Z}^2 \) satisfying \( a = bq_0 + r_0 \) and \( 0 \leq r_0 < b \). We have \(-b < r - r_0 < b\) so
\[
|r - r_0| < b.
\]
But \( |r - r_0| = |(a - bq) - (a_0 - bq_0)| = b|q_0 - q| \) where \( |q_0 - q| \in \mathbb{N} \) so
\[
|r - r_0| \in \{0, b, 2b, 3b, \ldots \}.
\]
But among these elements, only 0 is \( < b \), so in fact we have \(|r - r_0| = 0 \). Therefore \( r = r_0 \) and \( b(q - q_0) = 0 \) which implies \( q = q_0 \) since \( b \geq 1 \). We have proved that \((q, r) = (q_0, r_0)\). □