1 Well defined functions

If you look up on the internet the idea of “well defined functions” you will find many things, and some of them may sound a bit complicated. I just want to summarize what we think of in our context when we talk about well defined or not well defined functions.

Example 1. 1. The function

\[ f_1 : \mathbb{R} \rightarrow \mathbb{R} \]

\[ x \mapsto \frac{1}{x} \]

is not well defined because \( f_1(0) \) is not defined.

2. The function

\[ f_2 : \mathbb{R} \setminus \{0\} \rightarrow [0, +\infty) \]

\[ x \mapsto \frac{1}{x} \]

is not well defined because the target space is too small: for \( x = -\frac{1}{2} \) we have \( \frac{1}{x} = -2 \) which does not lie in the target space \([0, +\infty)\).

3. The function

\[ f_3 : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \]

\[ x \mapsto \frac{1}{x} \]

is well defined (and not surjective, but that’s another question).

4. The function

\[ f_4 : \mathbb{Z} \rightarrow \mathbb{Z} \]

\[ x \mapsto \begin{cases} 4x + 32 & \text{if } x \equiv 0 \text{ mod } 3 \\ 16x - 3 & \text{if } x \equiv 2 \text{ mod } 3 \end{cases} \]

is not well defined because I don’t know what the image of 4 is.

5. The function

\[ f_5 : \mathbb{Z} \rightarrow \mathbb{Z} \]

\[ x \mapsto \begin{cases} 4x + 32 & \text{if } x \equiv 0 \text{ mod } 3 \\ 16x - 3 & \text{if } x \text{ is even} \end{cases} \]

is not well defined because I cannot tell what the image of 6 would be!

Example 2. Problem 4 Of Lecture Notes on Cardinality. Let \( A, B, A', B' \) sets. Suppose that \( |A| = |A'| \) and \( |B| = |B'| \), and that \( A \cap B = \emptyset \) and \( A' \cap B' = \emptyset \). Show that \( |A \cup B| = |A' \cup B'| \).

Solution: Since we know \( |A| = |A'| \) and \( |B| = |B'| \), we know that there exist bijections

\[ f_A : A \rightarrow A' \]

and

\[ f_B : B \rightarrow B' \].
We define the function

\[ f : A \cup B \longrightarrow A' \cup B' \]

\[ x \longmapsto \begin{cases} 
  f_A(x) & \text{if } x \in A \\
  f_B(x) & \text{if } x \in B 
\end{cases} \]

This is a function that looks a bit like \( f_4 \) and \( f_5 \) above. It does not have the flaw of \( f_4 \) because every element of \( A \cup B \) is in \( A \) or in \( B \). It does not have the flaw of \( f_5 \) because \( A \cap B = \emptyset \). This is where we use this important hypothesis.

In fact \( f \) is well defined: we see that there is no problem, if I pick \( x \in A \cup B \), it is either in \( A \) or in \( B \) (and not in both) so I can compute its image (\( f_A(x) \) or \( f_B(x) \) accordingly) and this lands indeed in \( A' \cup B' \).

To prove that \( f \) is bijective, we provide its inverse function. We call \( f_A^{-1} : A' \rightarrow A \) and \( f_B^{-1} : B' \rightarrow B \) the inverse functions of \( f_A \) and \( f_B \) respectively, and we may define the function

\[ g : A' \cup B' \longrightarrow A \cup B' \]

\[ x \longmapsto \begin{cases} 
  f_A^{-1}(x) & \text{if } x \in A' \\
  f_B^{-1}(x) & \text{if } x \in B' 
\end{cases} \]

Again, here it is important that we know \( A' \cap B' = \emptyset \) for the function \( g \) to be well defined. Once we have stated that \( g \) is well defined, it is almost immediate to check that

\[ g \circ f = \text{id}_{A \cup B} \quad \text{and} \quad f \circ g = \text{id}_{A' \cup B'} . \]

This is the last bit that I “brushed off” a bit during the online session. I meant to say: this is the easy part that is just sheer verification of the formulas (\( g \circ f = \text{id}_{A \cup B} \) and \( f \circ g = \text{id}_{A' \cup B'} \)) while the subtle point of this problem was to notice how and when to use the hypotheses \( A \cap B = \emptyset \) and \( A' \cap B' = \emptyset \).