Problem 2

(1) $\mathcal{H} = \mathrm{End}_G(\underbrace{k[G/H]}_{=\mathrm{ind}_H^G(1)})$

By adjunction, we have isomorphisms of vector spaces

$$\begin{aligned}\mathcal{H} &= \mathrm{Hom}_G(\mathrm{ind}_H^G(1), \mathrm{ind}_H^G(1)) \\ &\overset{(1)}{\simeq} \mathrm{Hom}_H(k_{triv}, \mathrm{ind}_H^G(1)|_H) \\ &\overset{(2)}{\simeq} (\mathrm{ind}_H^G(1))^H = (k[G/H])^H = k[H\backslash G/H]\end{aligned}$$

Since we want to check that this identification is in fact an isomorphism of algebras. So we need to make it explicit.

Let $F \in \mathcal{H}$. By the isomorphism (1) above, F is taken onto the map

$$\begin{aligned} k_{triv} &\longrightarrow \mathrm{ind}_H^G(1) \\ 1 &\longmapsto F(1_H)\end{aligned}$$

Then by (2), it becomes the element $F(1_H)$.

So the identification is given by

$$\begin{aligned}\mathcal{H} &\xrightarrow{\sim} k[H\backslash G/H] \\ F &\longmapsto F(1_H)\end{aligned}$$

Let $E, F \in \mathcal{H}$ and $\varphi = E(1_H) \in k[{}_H\backslash G/_H]$

$\psi = F(1_H) \in k[{}_H\backslash G/_H]$

<u>Compute $E \circ F(1_H)$</u>

$$F(1_H) = \sum_{g \in G/H} \lambda_g \, 1_{gH} \quad (\text{seen in } k[G/H])$$

and then, since E is G-equivariant,

$$E(F(1_H)) = \sum_{g \in G/H} \lambda_g \, g \circ E(1_H) = \sum_{g \in G/H} \lambda_g \, g \circ \varphi$$

action of G on $k[G/H]$

Instead of λ_g, we may write $\psi(g)$

So $E \circ F(1_H) = \sum_{g \in G/H} \psi(g) \, g \circ \varphi$

For any $g, x \in G$ and $f \in k[G/H]$ we have, by definition

$$(g \circ f)(x) = f(g^{-1}x)$$

So for any $x \in G$ we have

$$\left(E \circ F(1_H)\right)(x) = \sum_{g \in G/H} \psi(g) \, \varphi(g^{-1}x)$$

(2) There is a natural functor

$$\mathrm{Rep}_k\, G \longrightarrow \mathrm{Vec}_k$$
$$V \longmapsto \mathrm{Hom}_G(\mathrm{ind}_H^G(1), V) \simeq V^H$$

In fact it has values in the category of right $\mathcal{H}$-modules since $\mathcal{H}$ obviously acts on the right on

$$\mathrm{Hom}_G(\mathrm{ind}_H^G(1), V)$$

(3) We talked about this in class.

(4) ~~Hom_G~~ Let M be a right $\mathcal{H}$-module and $M \otimes_{\mathcal{H}} k[G/H]$ the representation of G where G acts on $k[G/H]$

We want to check that for any $V \in \mathrm{Rep}_k(G)$:

$$\mathrm{Hom}_G(M \otimes_{\mathcal{H}} k[G/H], V) = \mathrm{Hom}_{\mathcal{H}}(M, V^H)$$

- Let $F: M \otimes_{\mathcal{H}} k[G/H] \longrightarrow V$ a G-equivariant map

Attach to it $f: M \longrightarrow V^H$
$$m \longmapsto F(m \otimes 1_H)$$

Check that it is right $\mathcal{H}$-equivariant.

Hint: For $V \in \mathrm{Rep}_k G$, first compute the action of $h \in \mathcal{H}$ on V^H:

To $v \in V^H$ attach the G-equiv map

$$\varphi_v : k[G/H] \longrightarrow V$$
$$1_H \longmapsto v$$

then $v \cdot h = \varphi_v \circ h(1_H)$

Note also that if $F: V \longrightarrow W$ is G-equivariant then

$$F\big|_{V^U}^{W^U} : V^U \longrightarrow W^U$$

is right $\mathcal{H}$-equivariant.

- Let $f: M \longrightarrow V^H$ and $m \in M$.

Attach to it the map

$$F_m : k[G/H] \longrightarrow V$$
$$1_H \longmapsto f(m)$$

(it is well defined by Frobenius adjunction)

Then build:

$$F : M \otimes_{\mathcal{H}} k[G/H] \longrightarrow V$$
$$m \otimes \varphi \longmapsto F_m(\varphi)$$

Check that it is well defined & G-equiv.

(5) —

(6) (b) $$BsB = \coprod_{a \in \mathbb{F}_q} Bs\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

Show that $1_{BsB} * 1_{BsB} = (q-1) 1_{BsB} + q 1_B$

(Note: it becomes

$$1_{BsB} * 1_{BsB} = - 1_{BsB}$$

in characteristic p!)