(1) A left \( \mathbb{Z}G \)-module \( M \) can be regarded as a right module by \( mg = g^{-1}m \). Therefore we can define the tensor product \( M \otimes_{\mathbb{Z}G} N \) of two \( \mathbb{Z}G \)-modules \( M, N \).

(a) Prove that \( M \otimes_{\mathbb{Z}G} N \) has the structure of a \( \mathbb{Z}G \)-module defined by the diagonal action \( g(m \otimes n) = gm \otimes gn \). Deduce that \( M \otimes_{\mathbb{Z}G} N = (M \otimes_{\mathbb{Z}} N)_G \), where \((-)_G\) denotes the coinvariant submodule.

(b) Prove that \( \text{Hom}_{\mathbb{Z}}(M, N) \) has the structure of a \( \mathbb{Z}G \)-module defined by the conjugation action \( (gf)(m) = gf(g^{-1}m) \). Deduce that \( \text{Hom}_{\mathbb{Z}G}(M, N) = \text{Hom}_{\mathbb{Z}}(M, N)^G \), where \((-)^G\) denotes the invariant submodule.

(c) Let \( P \) be a projective \( \mathbb{Z}G \)-module and \( M \) be a \( \mathbb{Z} \)-free \( \mathbb{Z}G \)-module. Prove that \( P \otimes_{\mathbb{Z}} M \) is \( \mathbb{Z}G \)-projective.

(2) Let \( G \) be a finite group and \( N : M \to M \) denote the norm map defined by multiplication by \( \sum_{g \in G} g \). Prove that the induced map \( \bar{N} : M_G \to M^G \) is an isomorphism if \( M \) is a projective \( \mathbb{Z}G \)-module.

(3) Let \( F \) be a right exact functor. Assume \( \cdots \to C_1 \to C_0 \to M \to 0 \) is exact where each \( C_i \) satisfies \( L_n F(C_i) = 0 \) for \( n > 0 \). Prove that \( L_n F(M) \cong H_n(F(C)) \) for all \( n \).

(4) Let \( R = \mathbb{Z}/m \) and \( M = \mathbb{Z}/d \) where \( d > 1 \) divides \( m \). Prove that \( M \) is not injective as an \( R \)-module if there exists a prime dividing both \( d \) and \( m/d \).

(5) Prove that the following are equivalent.

(a) \( N \) is an injective \( R \)-module.

(b) \( \text{Ext}^i(R, N) \) vanishes for all \( i > 0 \) and all \( M \).

(c) \( \text{Ext}^1(M, N) \) vanishes for all \( M \).

State and prove the dual statement for the projective case.

(6) Prove that

(a) \( \text{Tor}^R_i(A, \oplus_k B_k) \cong \oplus_k \text{Tor}^R_i(A, B_k) \).

(b) \( \text{Ext}^i_R(\oplus_k A_k, B) \cong \prod_k \text{Ext}^i_R(A_k, B) \).

(7) Let \( G \) be a finite group. Let \( P \to \mathbb{Z} \) denote a projective \( \mathbb{Z}G \)-resolution of the trivial module \( \mathbb{Z} \), and \( M \) a \( \mathbb{Z}G \)-module. Assume each \( P_i \) is finitely generated.

(a) Let \( Q^n = \text{Hom}_{\mathbb{Z}G}(P_n, \mathbb{Z}) \) for \( n \geq 0 \). Show that \( 0 \to \mathbb{Z} \to Q^0 \to Q^1 \to \cdots \) is exact and \( Q^n \) is \( \mathbb{Z}G \)-projective for all \( n \geq 0 \).
(b) By splicing $P \to Z$ and $Z \to Q$ obtain an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to Q^0 \to Q^1 \to Q^2 \to \cdots$$

of projective $\mathbb{Z}G$-modules. Set $F_i = P_i$ if $i \geq 0$ and $F_i = Q^{-i-1}$ if $i < 0$. The groups $\hat{H}^n(G, M) = H^n(\text{Hom}_{\mathbb{Z}G\text{-mod}}(F, M))$ are called the Tate cohomology groups of $G$. Prove that

$$\hat{H}^n(G, M) \cong \begin{cases} 
H^n(G, M) & \text{if } n > 0 \\
\text{Coker } \bar{N} & \text{if } n = 0 \\
\ker \bar{N} & \text{if } n = -1 \\
H_{-n-1}(G, M) & \text{if } n < -1 
\end{cases}$$

where $\bar{N}: M_G \to M^G$ is as in (2).