MATH 9144A - ASSIGNMENT 1

(1) Categories
(a) Prove that if a group $G$ is regarded as a category with one object then natural transformations on the identity functor are given by the central elements in $G$.
(b) Assume $C$ is complete. Prove that $\Delta \dashv \lim$ where $\Delta$ is the constant functor.
(c) Prove that $\text{Hom}_C(\prod_i A_i, B) \cong \prod_i \text{Hom}_C(A_i, B)$.
(d) Prove that in an additive category the natural map $A \coprod B \to A \prod B$ is an isomorphism. If $T : A \to B$ is an additive functor between additive categories then prove that the natural map $T(A) \oplus T(B) \to T(A \oplus B)$ is an isomorphism.

(2) Exactness
(a) Let $A$ be an abelian category. Prove that $\text{Hom}_A(M, -)$ and $\text{Hom}_A(-, M)$ are left exact.
(b) Consider the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ in the category of abelian groups. Give an example of $M$ such that $\text{Hom}(M, -)$ fails to be right exact. Similarly find an $N$ such that $\text{Hom}(-, N)$ fails to be right exact.
(c) Prove that $M \otimes_R -$ is right exact on $R$-mod.
(d) In the category of abelian groups find an exact sequence and an $M$ where $M \otimes -$ fails to be left exact.
(e) Give an example of a ring $R$ and an $R$-module $M$ such that $M \otimes_R -$ and $\text{Hom}(M, -)$ are both exact.

(3) Cone & Cylinder
(a) Prove that the two obvious inclusions $(1, 0, 0) : A \to \text{cyl}({\text{Id}_A})$ and $(0, 0, 1) : A \to \text{cyl}({\text{Id}_A})$ are both chain homotopy equivalences.
(b) Prove that $f : A \to B$ is null homotopic $s : f \simeq 0$ if and only if $f$ extends to $(-s, f) : \text{cone}({\text{Id}_A}) \to B$.

(4) Group homology
(a) Let $G = \langle t | t^n = 1 \rangle$ and $C$ denote the chain complex $\mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G$ (degree one and zero). Show that there is a long exact sequence
$$\cdots \to H_k(G) \to H_{k-2}(G) \to H_{k-1}(C) \to H_{k-1}(G) \to \cdots \to H_2(G) \to H_0(G) \to H_1(C) \to H_1(G) \to 0$$

1An element $z$ is central if $gz = zg$ for all $g \in G$.
2See Weibel for the definition of left/right-exactness.
and deduce that $H_k(G) \cong H_{k-2}(G)$ for $k > 2$.

*Hint:* Start with a map $f : C \to P$ where $P$ is the periodic free resolution of $G$ constructed in class. Now identify the cokernel.

(b) Prove that $H_1(G) = G/[G, G]$ by using the bar resolution $B$.

(c) Let $n$ denote the order of $G$. Prove that the map $H_k(G) \to H_k(G)$ given by multiplication by $n$ is the zero map.

*Hint:* Consider the chain map $N : B \to B$ given by multiplication by the norm element $N = \sum_{g \in G} g$. 