Mostly homology with coefficients

Due [Fri Mar 03].

More examples of spaces

1. Find a pair of path-connected spaces $X$ and $Y$ with $\pi_1(X) \cong \pi_1(Y) \neq 1$ but $H_*(X; \mathbb{Z}) \neq H_*(Y; \mathbb{Z})$.
2. Compute the integral homology of the $m$-skeleton of the $n$-simplex (this $m$-skeleton is the union of all faces of dimension at most $m$ of $\Delta^n$).

Homology with coefficients

3. a) Compute $H_*(\mathbb{R}P^n; A)$ for an arbitrary abelian group $A$ using cellular homology.
   b) Find an abelian group $A$ for which $\mathbb{R}P^n$ has the same $A$-homology as the point for all even $n$.
4. Let $f : S^n \to S^n$ be a map of degree $m$ and construct a space $X$ by attaching an $(n+1)$-cell to $S^n$ using $f$ as the attaching map.
   a) Compute $H_*(X; \mathbb{Z})$.
   b) Let $g : X \to X/S^n$ be the quotient map. Show $g$ induces the zero homomorphism on integral homology.
   c) Use homology with coefficients in $\mathbb{Z}/m$ to show that $g$ is not homotopic to a constant.
5. a) Let $p : Y \to X$ be a two-sheeted covering space. Show there is a short exact sequence of chain complexes
   $$0 \to C_*(X; \mathbb{Z}/2) \xrightarrow{\tau} C_*(Y; \mathbb{Z}/2) \xrightarrow{p_*} C_*(X; \mathbb{Z}/2) \to 0,$$
   where $\tau(\sigma)$ for a singular simplex $\sigma : \Delta^k \to X$ is defined to be the sum of all lifts $\bar{\sigma} : \Delta^k \to Y$, that is maps with $p \circ \bar{\sigma} = \sigma$.
   b) Use the long exact sequence in homology corresponding to that short exact sequence to recompute $H_*(\mathbb{R}P^n; \mathbb{Z}/2)$.
6. Is there a finite CW-complex $X$ such that $H_*(X; \mathbb{Z}/3) \cong H_*(S^4; \mathbb{Z}/3)$ and $H_*(X; \mathbb{Z}/2) \cong H_*(S^5; \mathbb{Z}/2)$?

Degree of maps

7. a) Show that for any map $f : S^{2n} \to S^{2n}$ there exists an $x$ with satisfying $f(x) = x$ or $f(x) = -x$. Why does that prove that any map $g : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point?
   b) Does every map $g : \mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ have a fixed point?

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\footnote{A finite CW-complex is just a CW-complex with finitely many cells.}
Homological algebra

8. a) Show that a commutative diagram of the form

\[
\begin{array}{cccccccc}
\cdots & \longrightarrow & C_{n+1} & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & E_{n+1} & \longrightarrow & A_n & \longrightarrow & D_n & \longrightarrow & E_n & \longrightarrow & A_{n-1} & \longrightarrow & D_{n-1} & \longrightarrow & \cdots \\
\end{array}
\]

where the top and bottom rows are long exact sequences, and the vertical equal signs denote the identity, gives rise to a long exact sequence

\[
\cdots \rightarrow E_{n+1} \rightarrow B_n \rightarrow C_n \oplus D_n \rightarrow E_n \rightarrow B_{n-1} \rightarrow \cdots
\]

where the maps are obtained from the diagram in an obvious way except that the map \(B_n \rightarrow C_n \oplus D_n\) has a minus sign in one coordinate.

b) Use this to give a new deduction of the Mayer-Vietoris sequence from the long exact sequences of the pairs \((A, A \cap B)\) and \((X, A)\), where \(X = \text{int}(A) \cup \text{int}(B)\).