PRACTICE SET 3

SECTION 6.1

Exercise 4. What is the remainder when $5!25!$ is divided by $31$?

Exercise 10. What is the remainder when $6^{2000}$ is divided by $11$?

Exercise 12. Using Fermat’s little theorem, find the least positive residue of $2^{1.000.000}$ modulo $17$.

Exercise 24. Show that $1^p + 2^p + 3^p + \cdots + (p-1)^p \equiv 0 \pmod{p}$ when $p$ is an odd prime.

SECTION 6.2

Exercise 2. Show that $45$ is a pseudoprime to the bases $17$ and $19$.

Exercise 8. Show that if $p$ is a prime and $2^p - 1$ is composite, then $2^p - 1$ is a pseudoprime to the base $2$.

Exercise 12. Show that $25$ is a strong pseudoprime to the base $7$.

Exercise 18. (a) Show that every integer of the form $(6m+1)(12m+1)(18m+1)$, where $m$ is a positive integer such that $6m+1, 12m+1,$ and $18m+1$ are all primes, is a Carmichael number.

(b) Conclude from part (a) that $1729 = 7 \cdot 13 \cdot 19$; $294,409 = 37 \cdot 73 \cdot 109$; $56,052,361 = 211 \cdot 421 \cdot 631$; $118,901,521 = 271 \cdot 541 \cdot 811$; and $172,947,529 = 307 \cdot 613 \cdot 919$ are Carmichael numbers.

SECTION 6.3

Exercise 6. Find the last digit of the decimal expansion of $\pi^{999.999}$.

Exercise 8. Show that if $a$ is an integer such that $a$ is not divisible by $3$ or such that $a$ is divisible by $9$, then $a^7 \equiv a \pmod{63}$.

Exercise 10. Show that $a^{\phi(b)} + b^{\phi(a)} \equiv 1 \pmod{ab}$, if $a$ and $b$ are relatively prime positive integers.

Exercise 14. Show that the solutions to the simultaneous system of congruences

\[
\begin{align*}
    x &\equiv a_1 \pmod{m_1} \\
    x &\equiv a_2 \pmod{m_2} \\
    \vdots \\
    x &\equiv a_r \pmod{m_r},
\end{align*}
\]
where the \( m_j \) are pairwise relatively prime, are given by

\[ x \equiv a_1 M_1^{\phi(m_1)} + a_2 M_2^{\phi(m_2)} + \cdots + a_r M_r^{\phi(m_r)} \pmod{M}, \]

where \( M = m_1 m_2 \cdots m_r \) and \( M_j = M/m_j \) for \( j = 1, 2, \ldots, r \).

**Section 7.1**

**Exercise 4.** Find all positive integers \( n \) such that \( \phi(n) \) has each of these values. Be sure to prove that you have found all solutions.

(a) 1  
(b) 2  
(c) 3  
(d) 4

**Exercise 8.** Show that there is no positive integer \( n \) such that \( \phi(n) = 14 \).

**Exercise 18.** Show that if \( n \) is an odd integer, then \( \phi(4n) = 2\phi(n) \).