1. Section 1.3

Problem 4. We see that
\[
\frac{1}{1\cdot 2} = \frac{1}{2}, \quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{2}{3}, \quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4},
\]
and is reasonable to conjecture
\[
\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.
\]

We will prove this formula by induction.

Base $n = 1$: It is shown above.

Hypothesis: Suppose the formula holds for $n$.

Step:
\[
\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)}
= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}
= \frac{n(n+2) + 1}{(n+1)(n+2)} = \frac{n^2 + 2n + 1}{(n+1)(n+2)}
= \frac{(n+2)^2}{(n+1)(n+2)} = \frac{n+1}{n+2},
\]
where in the second equality we used the induction hypothesis.

Problem 14. We will use strong induction.

Base $54 \leq n \leq 60$: We have
\[
54 = 7\cdot 2 + 10\cdot 4, \quad 55 = 7\cdot 5 + 10\cdot 2, \quad 56 = 7\cdot 8 + 10\cdot 0, \quad 57 = 7\cdot 1 + 10\cdot 5
\]
and
\[
58 = 7\cdot 4 + 10\cdot 3, \quad 59 = 7\cdot 7 + 10\cdot 1, \quad 60 = 7\cdot 0 + 10\cdot 6.
\]

Hypothesis: Suppose the result holds for $54 \leq k \leq n$.

Step $n \geq 60$: We have $n - 6 \geq 54$, hence by the induction hypothesis we can write
\[
n - 6 = 7a + 10b \quad \text{for some } a, b \in \mathbb{Z}_{>0}.
\]
Then $n + 1 = 7(a + 1) + 10b$, as desired.

\[\text{Date: September 27, 2016.}\]
Problem 22. We will use induction.

Base $n = 0$: We have $1 + 0h = 1 = (1 + h)^0$, as desired.

Hypothesis: Suppose the result holds for $n$.

Step $n \geq 0$: We have

$$
(1 + h)^{n+1} = (1 + h)^n (1 + h)
\geq (1 + nh)(1 + h)
= 1 + h + nh + nh^2
\geq 1 + (n + 1)h,
$$

where in the first inequality we used the induction hypothesis and $1 + h \geq 0$.

Problem 24. The proof fails in the statement that the sets $\{1, \ldots, n\}$ and $\{2, \ldots, n+1\}$ have common members. This is false when $n = 1$; indeed, the sets are $\{1\}$ and $\{2\}$ which are clearly disjoint.

2. Section 1.5

Problem 26. Let $a, b \in \mathbb{Z}_{>0}$.

We first prove existence. The division algorithm gives $q', r' \in \mathbb{Z}$ such that

$$a = bq' + r' \quad \text{with} \quad 0 \leq r' < b.$$  

We now divide into two cases:

(i) Suppose $r' \leq b/2$; then $-b/2 < r' \leq b/2$. The result follows by taking $q = q'$ and $r = r'$.

(ii) Suppose $b/2 < r' < b$; then $-b/2 < r' - b < 0$. We have

$$a = bq' + r' = bq' + b + r' - b = b(q' + 1) + (r' - b),$$

Write $q = q' + 1$ and $r = r' - b$. Then

$$a = bq + r, \quad \text{with} \quad -b/2 < r < 0 \leq b/2.$$

as desired.

We now prove uniqueness. Suppose

$$a = bq_1 + r_1 = bq_2 + r_2, \quad \text{with} \quad -b/2 < r_1, r_2 \leq b/2.$$  

Then $b(q_1 - q_2) = (r_2 - r_1)$ and $b$ divides $r_2 - r_1$. Since $-b < r_2 - r_1 < b$ it follows that $r_2 - r_1 = 0$ because there is no other multiple of $b$ in this interval. We conclude that $r_1 = r_2$ and $b(q_1 - q_2) = 0$; thus we also have $q_1 = q_2$, as desired.

Problem 36. Let $a \in \mathbb{Z}$. Dividing $a$ by 3 we get $a = 3q + r$ with $r = 0, 1, 2$. Note that

$$a^3 - a = (a - 1)a(a + 1) = (3q + r - 1)(3q + r)(3q + r + 1)$$

and clearly for any choice of $r = 0, 1, 2$ one of the three factors is a multiple of 3. This is the same as saying that in among three consecutive integers one must be a multiple of 3.
3. Section 2.1

Problem 12. Let \( a \in \mathbb{Z}_{>0} \).

We first prove existence. We will use strong induction.

*Base* \( a \leq 2 \). If \( a = 1 \) take \( k = 0 \) and \( e_0 = 1 \); if \( a = 2 \) take \( k = 1 \), \( e_1 = 1 \) and \( e_0 = -1 \).

*Hypothesis:* Suppose the desired expression exists for all positive integers \( < a \).

*Step* \( a \geq 3 \). From the modified division algorithm (Problem 26 in Section 1.5) there exist \( q, e_0 \in \mathbb{Z} \) such that

\[
a = 3q + r, \quad \text{with} \quad -3/2 < r \leq 3/2;
\]

in particular, \( r = -1, 0, 1 \). We have \( 0 < q = (a - r)/3 < a \) and by hypothesis we can write

\[
q = a_s 3^s + 3 + a_1 3 + a_0, \quad a_s \neq 0, \quad a_i \in \{-1, 0, 1\}.
\]

Thus we have

\[
a = 3q + r = 3(a_s 3^s + \ldots + a_1 3 + a_0) + r = a_s 3^{s+1} + \ldots + a_1 3^2 + a_0 3 + r
\]

and we take \( k = s + 1 \), \( e_0 = r \) and \( e_i = a_{s-i} \) for \( i = 1, \ldots, k \).

We now prove uniqueness. We will use strong induction. Suppose

\[
a = e_k 3^k + \ldots + e_1 3 + e_0 = c_s 3^s + \ldots + c_1 3 + c_0, \quad e_k, a_s \neq 0, \quad e_i, a_i \in \{-1, 0, 1\}.
\]

*Base* \( a \leq 2 \): We know from above that if \( a = 1 \) can we take \( k = 0 \) and \( e_0 = 1 \) and if \( a = 2 \) we can take \( k = 1 \), \( e_1 = 1 \) and \( e_0 = -1 \), as balanced ternary expansions. Note also that 0 cannot be written as an expansion using non-zero coefficients.

Suppose now \( a = 1 = e_k 3^k + \ldots + e_1 3 + e_0 \) with \( k \geq 1 \); then \( a \) divided by 3 has reminder \( e_0 = 1 \) by the division algorithm. We conclude that \( e_k 3^k + \ldots + e_1 3 = 0 \) which is impossible.

Suppose \( a = 2 = 1 \cdot 3 - 1 = e_k 3^k + \ldots + e_1 3 + e_0 \) with \( k \geq 1 \); then \( a \) divided by 3 has reminder \( e_0 = -1 \) by the modified division algorithm. We conclude that \( e_k 3^k + \ldots + e_1 3 = 3 \). Dividing both sides by 3 we conclude that \( e_k 3^{k-1} + \ldots + e_1 = 1 \) which gives \( k = 1 \) and \( e_1 = 1 \) by the previous paragraph. This shows that \( a = 1, 2 \) have an unique balanced ternary expansion.

*Hypothesis:* Suppose the expansion is unique for all positive integers \( < a \).

*Step* \( a \geq 3 \): By the uniqueness of the modified division algorithm (Problem 26, Section 1.5), dividing \( a \) by 3 we conclude \( e_0 = c_0 \). Now

\[
\frac{a - e_0}{3} = e_k 3^{k-1} + \ldots + e_1 = c_s 3^{s-1} + \ldots + c_1
\]

and by induction hypothesis we have \( k = s \) and \( e_i = c_i \) for \( i = 1, \ldots, k \).

Finally, suppose \( a < 0 \); we apply the result to \(-a > 0 \) and (due to the symmetry of the coefficients) we obtain the expansion for \( a \) by multiplying by \(-1\) the expansion for \(-a\).

Problem 13. Let \( w \) be the weight to be measured. From the previous exercise we can write

\[
w = e_k 3^k + \ldots + e_1 3 + e_0, \quad e_k \neq 0, \quad e_i \in \{-1, 0, 1\}.
\]

Place the object in pan 1. If \( e_i = 1 \), then place a weight of \( 3^i \) into pan 2; if \( e_i = -1 \), then place a weight of \( 3^i \) into pan 1; if \( e_i = 0 \) do nothing; in the end the pans are balanced.
Problem 17. Let \( n \in \mathbb{Z}_{>0} \) be given in base \( b \) by
\[
    n = a_k b^k + \ldots + a_1 b + a_0, \quad a_k \neq 0, \quad 0 \leq a_i < b.
\]
Let \( m \in \mathbb{Z}_{>0} \). We want to find the base \( b \) expansion of \( b^m n \), that is
\[
    b^m n = c_s b^s + \ldots + c_1 b + c_0, \quad c_s \neq 0, \quad 0 \leq c_i < b.
\]
Multiplying both sides of the first equation by \( b^m \) gives
\[
    b^m n = a_k b^{k+m} + \ldots + a_1 b^{m+1} + a_0 b^m, \quad a_k \neq 0, \quad 0 \leq a_i < b.
\]
We know that the expansion in base \( b \) is unique, so by comparing the last two equations we conclude that
\[
    s = k + m, \quad c_{s-i} = a_{k-i} \text{ for } i = 0, \ldots, k \quad \text{and} \quad c_i = 0 \text{ for } i = 0, \ldots, m-1,
\]
which means
\[
    b^m n = (c_s c_{s-1} \ldots c_0)_b = (a_k a_{k-1} \ldots a_1 a_0 00 \ldots 0)_b,
\]
where we have \( m \) zeros in the end.

4. Section 3.1

Problem 6. Let \( n \in \mathbb{Z} \). Note the factorization \( n^3 + 1 = (n+1)(n^2 - n + 1) \) into two integers. If \( n^3 + 1 \) is a prime, then \( n \geq 1 \) and \( n + 1 \) is either 1 or prime. Since \( n + 1 \neq 1 \) we have \( n + 1 \) is prime and hence \( n^2 - n + 1 \) must be 1, which implies \( n = 0, 1 \). We conclude \( n = 1 \), as desired.

Problem 8. Let \( n \geq 1 \in \mathbb{Z}_{>0} \). Consider \( Q_n = n! + 1 \). There is a prime factor \( p \mid Q_n \). Suppose \( p \leq n \); then \( p \mid Q_n - n! = 1 \), a contradiction. We conclude that \( p > n \). In particular, given a positive integer \( n \) we can always find a prime larger than \( n \); by growing \( n \) we produce infinitely many arbitrarily large primes.

5. Section 3.3

Problem 6. Let \( a \in \mathbb{Z}_{>0} \). Suppose \( d \) is a common divisor of \( a \) and \( a + 2 \). Then \( d \) divides \( a + 2 - a = 2 \), that is \( d = 1 \) or \( d = 2 \). Clearly, if \( a \) is odd then \( d = 1 \) and if \( a \) is even then \( 2 \mid a + 2 \). We conclude that \( (a, a + 2) = 1 \) if and only if \( a \) is odd and \( (a, a + 2) = 2 \) if and only if \( a \) is even.

Problem 10. Let \( a, b \in \mathbb{Z} \) satisfy \( (a, b) = 1 \). There exist \( x, y \in \mathbb{Z} \) such that \( ax + by = 1 \). Then
\[
    (a + b)(x + y) + (a - b)(x - y) = 2ax + 2by = 2(ax + by) = 2
\]
and since \( (a+b, a-b) \) is the smallest positive integer that can be written as an integral linear combination of \( a + b \) and \( a - b \) we must have \( (a+b, a-b) \leq 2 \). Thus \( (a+b, a-b) = 1, 2 \) as desired.

Here is an alternative, longer but more direct proof:
Write \( d = (a+b, a-b) \). If \( d = 1 \) there is nothing to prove. Suppose \( d \neq 1 \) and let \( p \) be a prime divisor of \( d \) (which exists because \( d \neq 1 \)). In particular, \( p \) is a common divisor of \( a+b \) and \( a-b \), therefore it divides both their sum and difference; more precisely, \( p \) divides
\[
    (a + b) + (a - b) = 2a \quad \text{and} \quad (a + b) - (a - b) = 2b.
\]
Furthermore, since \( p \) is prime we also have

(i) \( p \mid 2a \) implies \( p = 2 \) or \( p \mid a \),
(ii) \( p \mid 2b \) implies \( p = 2 \) or \( p \mid b \).

Suppose \( p \neq 2 \). Then in (i) we have \( p \mid a \) and in (ii) we have \( p \mid b \); this is a contradiction with \((a, b) = 1\). We conclude that \( p = 2 \).

So far we have shown that the unique prime factor of \( d \) is 2, therefore \( d = 2^k \) with \( k \geq 1 \). To finish the proof we need to prove that \( k = 1 \). Since \( d \mid a + b \) and \( d \mid a - b \) arguing as above we conclude that \( 2^k \mid 2a \) and \( 2^k \mid 2b \), that is

\[
2a = 2^k x \quad \text{and} \quad 2b = 2^k y \quad \text{for some} \quad x, y \in \mathbb{Z}.
\]

Suppose \( k \geq 2 \). Then dividing both equations by 2 we get

\[
a = 2^{k-1} x \quad \text{and} \quad b = 2^{k-1} y
\]

with \( k - 1 \geq 1 \). In particular \( 2 \mid a \) and \( 2 \mid b \), a contradiction with \((a, b) = 1\), showing that \( k = 1 \), as desired.

**Problem 12.** Let \( a, b \in \mathbb{Z} \) be even and not both zero. There exist \( x, y \in \mathbb{Z} \) such that

\[
ax + by = (a, b) \iff \frac{a}{2} x + \frac{b}{2} y = \frac{(a, b)}{2}.
\]

Since \((a/2, b/2)\) is the smallest positive integer that can be written as an integral linear combination of \( a/2 \) and \( b/2 \) we must have \((a/2, b/2) \leq (a, b)/2\).

To finish the proof we will show that \((a/2, b/2) \geq (a, b)/2\). There exist \( x, y \in \mathbb{Z} \) such that

\[
\frac{a}{2} x + \frac{b}{2} y = (a/2, b/2) \iff ax + by = 2(a/2, b/2).
\]

Since \((a, b)\) is the smallest positive integer that can be written as an integral linear combination of \( a \) and \( b \) we conclude \((a/2, b/2) \geq (a, b)/2\), as desired.

**Problem 24.** Let \( k \in \mathbb{Z}_{\geq 0} \). Suppose \( d \) is a common divisor of \( 3k + 2 \) and \( 5k + 3 \). Then \( d \) divides every integral linear combination of these numbers. In particular, \( d \) divides

\[
5(3k + 2) - 3(5k + 3) = 15k + 10 - 15k - 9 = 1,
\]

hence \((3k + 2, 5k + 3) = 1\), as desired.

6. Section 3.4

**Problem 2.** We will use the Euclidean algorithm.

a) **Compute** \((51, 87)\).

\[
87 = 51 \cdot 1 + 36, \quad 51 = 36 \cdot 1 + 15, \quad 36 = 15 \cdot 2 + 6, \quad 15 = 6 \cdot 2 + 3, \quad 6 = 3 \cdot 2 + 0,
\]

thus \((51, 87) = 3\).

b) **Compute** \((105, 300)\).

\[
300 = 105 \cdot 2 + 90, \quad 105 = 90 \cdot 1 + 15, \quad 90 = 15 \cdot 6 + 0,
\]

thus \((105, 300) = 15\).
c) Compute $(981, 1234)$.

\[ 1234 = 981 \cdot 1 + 253, \quad 981 = 253 \cdot 3 + 222, \quad 253 = 222 \cdot 1 + 31 \]

and

\[ 222 = 31 \cdot 7 + 5, \quad 31 = 5 \cdot 6 + 1, \quad 5 = 1 \cdot 5 + 0, \]

thus $(981, 1234) = 1$.

**Problem 6.**

a) Compute $(15, 35, 90)$.

Note that $90 = 15 \cdot 6$ then \(((15, 90), 35) = (15, 35) = 5$.

b) Compute $(300, 2160, 5040)$.

Note that $1260 = 300 \cdot 7 + 60$ and $300 = 60 \cdot 5$ thus $(300, 2160) = 60$.

Since $5040 = 60 \cdot 84$ we also have

\[ (300, 2160, 5040) = \left( (300, 2160), 5040 \right) = (60, 5040) = 60. \]

7. **Section 3.5**

**Problem 10.** Let $a, b \in \mathbb{Z}_{>0}$. Suppose $a^3 \mid b^2$.

Write $a = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ for the prime factorization of $a$. Write $p_i^{b_i}$ for the largest power of $p_i$ dividing $b$. In particular, we can write $b = p_i^{b_i} \cdot m$ for some $m \in \mathbb{Z}$, with $p_i \mid m$.

From $a^3 \mid b^2$ it follows that $p_i^{3a_i} \mid p_i^{2b_i} m^2$ and since $p_i \mid m$ we must have $p_i^{3a_i} \mid p_i^{2b_i}$. This implies $2b_i - 3a_i \geq 0$, hence $b_i/a_i \geq 3/2 > 1$. Thus $b_i > a_i$ for all $i$. Hence we can write

\[ b = p_1^{a_1} p_1^{b_1-a_1} \cdot p_2^{a_2} p_2^{b_2-a_2} \cdot \ldots \cdot p_k^{a_k} p_k^{b_k-a_k} \cdot m' \]

for some $m' \in \mathbb{Z}$ (note that $m'$ is needed since $b$ may have prime factors which are none of the $p_i$). Therefore, by reordering the factors we also have

\[ b = (p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k})(p_1^{b_1-a_1} p_2^{b_2-a_2} \ldots p_k^{b_k-a_k}) \cdot m' = a(p_1^{b_1-a_1} p_2^{b_2-a_2} \ldots p_k^{b_k-a_k}) \cdot m'. \]

Thus $a \mid b$, as desired.

**Problem 30.** We will use the formulas for $(a, b)$ and LCM$(a, b)$ in terms of the prime factorizations of $a$ and $b$.

a) $a = 2 \cdot 3^2 \cdot 5^3$, $b = 2^2 \cdot 3^3 \cdot 7^2$. Thus

\[ (a, b) = 2 \cdot 3^2, \quad \text{LCM}(a, b) = 2^2 \cdot 3^3 \cdot 5^3 \cdot 7^2. \]

b) $a = 2 \cdot 3 \cdot 5 \cdot 7$, $b = 7 \cdot 11 \cdot 13$. Thus

\[ (a, b) = 7, \quad \text{LCM}(a, b) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13. \]

c) $a = 2^8 \cdot 3^6 \cdot 5^4 \cdot 11^{13}$, $b = 2 \cdot 3 \cdot 5 \cdot 11 \cdot 13$. Thus

\[ (a, b) = 2 \cdot 3 \cdot 5 \cdot 11, \quad \text{LCM}(a, b) = 2^8 \cdot 3^6 \cdot 5^4 \cdot 11^{13} \cdot 13. \]
d) \[ a = 41^{101} \cdot 47^{43} \cdot 103^{1001}, \quad b = 41^{11} \cdot 43^{47} \cdot 83^{111}. \] Thus
\[ (a, b) = 41^{11}, \quad \text{LCM}(a, b) = 41^{101} \cdot 43^{47} \cdot 47^{43} \cdot 83^{111} \cdot 103^{1001}. \]

**Problem 34.** Let \( a, b \in \mathbb{Z}_{>0} \). Suppose that
\[ (a, b) = 18 = 2 \cdot 3^2 \quad \text{and} \quad \text{LCM}(a, b) = 540 = 2^3 \cdot 3^3 \cdot 5. \]
Since \((a, b) \cdot \text{LCM}(a, b) = ab\) we conclude that the possible prime factors of \( a, b \) are 2, 3 and 5. Write
\[ a = 2^{d_2}3^{d_3}5^{d_5}, \quad b = 2^{e_2}3^{e_3}5^{e_5}, \quad d_i, e_i \geq 0 \]
for the prime factorizations of \( a \) and \( b \). We also know that
\[ (a, b) = 2^{\min(d_2, e_2)} \cdot 3^{\min(d_3, e_3)} \cdot 5^{\min(d_5, e_5)} \]
and
\[ \text{LCM}(a, b) = 2^{\max(d_2, e_2)} \cdot 3^{\max(d_3, e_3)} \cdot 5^{\max(d_5, e_5)}. \]
Therefore,
\[ \min(d_2, e_2) = 1 \quad \max(d_2, e_2) = 2. \]
After interchanging \( a, b \) if necessary we can suppose \( d_2 = 1 \) and \( e_2 = 2 \). Similarly, we also have
\[ \min(d_3, e_3) = 2, \quad \max(d_3, e_3) = 3, \quad \min(d_5, e_5) = 0, \quad \max(d_5, e_5) = 1. \]
Thus \((d_3, e_3) = (2, 3)\) or \((3, 2)\) and \((d_5, e_5) = (1, 0)\) or \((1, 0)\), giving the following four possibilities for \( a, b \):

1. \[ a = 2^1 \cdot 3^2 = 18 \quad \text{and} \quad b = 2^2 \cdot 3^1 \cdot 5^1 = 540, \]
2. \[ a = 2^1 \cdot 3^2 \cdot 5^1 = 90 \quad \text{and} \quad b = 2^2 \cdot 3^3 = 108, \]
3. \[ a = 2^1 \cdot 3^3 = 54 \quad \text{and} \quad b = 2^2 \cdot 3^2 \cdot 5^1 = 180, \]
4. \[ a = 2^1 \cdot 3^3 \cdot 5^1 = 270 \quad \text{and} \quad b = 2^2 \cdot 3^2 = 36, \]

Since \((a, b)\) and \(\text{LCM}(a, b)\) do not depend on the signs and order of \( a, b \) we obtain all the solutions by multiplying \( a \) or \( b \) or both by \(-1\) and interchanging them: \((\pm 18, \pm 540), (\pm 540, \pm 18), (\pm 90, \pm 108), (\pm 108, \pm 90), (\pm 54, \pm 180), (\pm 180, \pm 54), (\pm 270, \pm 36), (\pm 36, \pm 270)\).

**Problem 56.** We will work by contradiction.

Suppose there are only finitely many primes of the form \(6k+5\). Denote them \( p_0 = 5, p_1, \ldots, p_k \) and consider the number
\[ N = 6p_0p_1 \cdots p_k - 1. \]
Clearly \( N > 1 \) because \( p_0 = 5 \). Let \( p \) be a prime dividing \( N \). We apply the division algorithm to divide \( p \) by 6 and obtain
\[ p = 6q + r, \quad r, q \in \mathbb{Z}, \quad 0 \leq r \leq 5. \]
We now divide into cases

1. Suppose \( r = 0, 2, 4 \); then \( p \) is even, i.e \( p = 2 \). Since \( 2 \nmid N \) (it divides \( N + 1 \)) this is impossible; thus \( r \neq 0, 2, 4 \).
2. Suppose \( r = 3 \); then \( 3 \mid p \), i.e \( p = 3 \). Again, \( 3 \nmid N \), a contradiction.
3. Suppose \( r = 5 \); thus \( p \) is of the form \( 6k+5 \) and by hypothesis we have \( p = p_i \) for some \( i \). Since \( p_i \mid N + 1 \) it does not divide \( N \), again a contradiction.
From the cases above it follows that all the prime factors of \( N \) are of the form \( 6k + 1 \).

Note that \((6k_1 + 1)(6k_2 + 1) = 6(6k_1k_2 + k_1 + k_2) + 1\), that is the product of integers of the form \( 6k + 1 \) is also of this form. Since we showed all the prime factors of \( N \) have this form we conclude that \( N \) is of the form \( 6k + 1 \) which is incompatible with \( N \) being also of the form \( 6k + 5 \) as defined above. Thus our initial assumption is wrong, i.e. there are infinitely many primes of the form \( 6k + 5 \), as desired.

\section{3.7}

\textbf{Problem 2.} We apply the theorem we learned in class to describe solutions of linear Diophantine equations.

\textbf{a) The equation} \( 3x + 4y = 7 \). Since \((3, 4) = 1 \mid 7 \) there are infinitely many solutions; note that \( x_0 = y_0 = 1 \) is a particular solution. Then all the solutions are of the form

\[ x = 1 + 4t, \quad y = 1 - 3t, \quad t \in \mathbb{Z}. \]

\textbf{b) The equation} \( 12x + 18y = 50 \). Since \((12, 18) = 6 \mid 50 \) there are no solutions.

\textbf{c) The equation} \( 30x + 47y = -11 \). Clearly \((30, 47) = 1 \) (47 is prime) so there are solutions. We find a particular solution by applying the Euclidean algorithm followed by back substitution. Indeed,

\[ 47 = 30 \cdot 1 + 17, \quad 30 = 17 \cdot 1 + 13, \quad 17 = 13 \cdot 1 + 4 \]

and

\[ 13 = 4 \cdot 3 + 1, \quad 4 = 1 \cdot 4 + 0; \]

in particular, this double-checks that \((30, 47) = 1\); we continue

\[ 1 = 13 - 4 \cdot 3 = 13 - (17 - 13) \cdot 3 = 13 \cdot 4 - 17 \cdot 3 = (30 - 17) \cdot 4 - 17 \cdot 3 = \]

\[ = 30 \cdot 4 - 17 \cdot 7 = 30 \cdot 4 - (47 - 30) \cdot 7 = 30 \cdot 11 - 47 \cdot 7. \]

Thus \( x_1 = 11, y_1 = -7 \) is a particular solution to \( 30x + 47y = 1 \). Thus \( x_0 = -11x_1 = -121, y_0 = -11y_1 = 77 \) is a particular solution to the desired equation. Therefore, the general solution is given by

\[ x = -121 + 47t, \quad y = 77 - 30t, \quad t \in \mathbb{Z}. \]

\textbf{d) The equation} \( 25x + 95y = 970 \). Since \((25, 95) = 5 \mid 970 \) there are infinitely many solutions. We divide both sides of the equation by 5 to obtain the equivalent equation

\[ 5x + 19y = 194. \]

Note that \((5, 19) = 1 \) and \( x_1 = 4, y_1 = -1 \) is a particular solution to \( 5x + 19y = 1 \); then \( x_0 = 194x_1 = 776, y_0 = 194y_1 = -194 \) is a particular solution to our equation. Thus the general solution is given by

\[ x = 776 + 19t, \quad y = -194 - 5t, \quad t \in \mathbb{Z}. \]

\textbf{d) The equation} \( 102x + 1001y = 1 \). We find \((102, 1001)\) by applying the Euclidean algorithm:

\[ 1001 = 102 \cdot 9 + 83, \quad 102 = 83 \cdot 1 + 19, \quad 83 = 19 \cdot 4 + 7 \]

and

\[ 19 = 7 \cdot 2 + 5, \quad 7 = 5 \cdot 1 + 2, \quad 5 = 2 \cdot 2 + 1, \quad \]
hence \((102,1001) = 1\) and the equation has infinitely many solutions. We apply back substitution to find a particular solution:

\[
1 = 5 - 2 \cdot 2 = 5 - (7 - 5) \cdot 2 = 7 \cdot (-2) - 5 \cdot 3 = 7 \cdot (-2) + (19 - 7 \cdot 2) \cdot 3 \\
= 19 \cdot 3 - 7 \cdot 8 = 19 \cdot 3 - (83 - 19 \cdot 4) \cdot 8 = 83 \cdot (-8) + 19 \cdot 35 \\
= 83 \cdot (-8) + (102 - 83) \cdot 35 = 102 \cdot 35 - 83 \cdot 43 = 102 \cdot 35 - (1001 - 102 \cdot 9) \cdot 43 \\
= 1001 \cdot (-43) + 102 \cdot 422.
\]

Thus \(x_0 = 422, y_0 = -43\) is a particular solution. Therefore, the general solution is given by

\[
x = 422 + 1001t, \quad y = -43 - 102t, \quad t \in \mathbb{Z}.
\]

**Problem 6.** This problem can be stated as finding a non-negative solution to the Diophantine equation \(63x + 7 = 23y\), where \(x\) is the number of plantains in a pile, and \(y\) is the number of plantains each traveler receives.

Replace \(y\) by \(-y\) and rearrange the equation into \(63x + 23y = -7\) and note that \((63,23) = 1\), hence there are infinitely many solutions. We apply Euclidean algorithm

\[
63 = 23 \cdot 2 + 17, \quad 23 = 17 \cdot 1 + 6, \quad 17 = 6 \cdot 2 + 5, \quad 6 = 5 \cdot 1 + 1
\]

and back substitution

\[
1 = 6 - 5 = 6 - (17 - 6 \cdot 2) = 6 \cdot 3 - 17 = (23 - 17) \cdot 3 - 17 = \\
= 23 \cdot 3 - 17 \cdot 4 = 23 \cdot 3 - (63 - 23 \cdot 2) \cdot 4 = 63 \cdot (-4) + 23 \cdot 11,
\]

hence \(x_1 = -4, y_0 = 11\) is a particular solution to \(63x + 23y = 1\). We conclude that \(x_0 = -7x_1 = 28, y_0 = -7y_1 = -77\) is a particular solution. Thus the general solution is given by

\[
x = 28 + 23t, \quad y = -77 - 63t, \quad t \in \mathbb{Z}.
\]

Replacing again \(y\) by \(-y\) we get the general solution to \(63x + 7 = 23y\) given by

\[
x = 28 + 23t, \quad y = 77 + 63t, \quad t \in \mathbb{Z}.
\]

These values of \(x, y\) are both positive when \(t \geq -1\), therefore the number of plantains in the pile could be any integer of the form \(28 + 23t\) for \(t \geq -1\).