Problem 1: Use mathematical induction to show that $7^n \equiv 1 + 6n \pmod{36}$ for every $n \geq 1$.

Solution: Base case. Set $n = 1$. Then $7^1 = 7 = 1 + 6 \cdot 1$, and our congruence holds.

For the induction step, assume that $7^n \equiv 1 + 6n \pmod{36}$. Then

$$7^{n+1} \equiv 7(7^n + 1) \equiv 7(1 + 6n) \equiv 7 + 42n \equiv 7 + 6(n + 1) \pmod{36}$$

Thus the desired congruence holds for $n + 1$. This completes the proof.

Problem 2: (5 marks) How many ways are there of paying exactly $4$ worth of postage by using a combination of 5-cent and 7-cent stamps? Explain your answer.

Solution: The problem is equivalent to finding the number of non-negative integer solutions to the Diophantine equation

$$5x + 7y = 400.$$

Let us first find all integer solutions, then identify the ones with $x, y \geq 0$ among them.

By trial and error, we see that $x_0 = 3$ and $y_0 = -2$ is a particular integer solution to $5x + 7y = 1$. (We can also use the Euclidean algorithm to find a particular solution at this point.)

Multiplying $x_0$ and $y_0$ by 400, we obtain a particular solution to $5x + 7y = 400$,

$$x_1 = 1200 \text{ and } y_1 = -800.$$

The general solution to $5x + 7y = 400$ is thus

$$x = 1200 + 7t \text{ and } y = -800 - 5t,$$

where $t$ ranges over the integers.

Let us now find which values of $t$ give us non-negative solutions. Setting $x \geq 0$, we obtain $1200 + 7t \geq 0$ or equivalently, $t \geq -\frac{1200}{7} = -171 \frac{3}{7}$.

Setting $y \geq 0$, we obtain $-800 - 5t \geq 0$ or equivalently, $t \leq -160$. Thus $t$ can be any integer between $-171$ and $-160$. There are exactly 12 such integers.

Problem 3: (6 marks) Find all pairs of positive integers $x, y$, such that $x > y$, the greatest common divisor of $x$ and $y$ is 30, and the least common multiple of $x$ and $y$ is 420. Explain your answer.

Solution: First note that 30 = $2 \cdot 3 \cdot 5$ and 420 = $2^2 \cdot 3 \cdot 5 \cdot 7$. Thus $x = 2^{a_2}3^{a_3}5^{a_5}7^{a_7}$ and $y = 2^{b_2}3^{b_3}5^{b_5}7^{b_7}$, where

$$\{a_2, b_2\} = \{1, 2\}, \quad a_3 = b_3 = 2, \quad a_5 = b_5 = 1,$$

$$\{a_7, b_7\} = \{0, 1\}.$$  

This leaves us with 4 possibilities:
(i) $a_2 = 1, b_2 = 2, a_7 = 0, b_7 = 1$, i.e., $x = 2 \cdot 3 \cdot 5 = 30$ and $y = 2^2 \cdot 3 \cdot 5 \cdot 7 = 420$.
(ii) $a_2 = 1, b_2 = 2, a_7 = 1, b_7 = 0$, i.e., $x = 2 \cdot 3 \cdot 5 \cdot 7 = 210$ and $y = 2^2 \cdot 3 \cdot 5 = 60$.
(iii) $a_2 = 2, b_2 = 1, a_7 = 0, b_7 = 1$, i.e., same as (ii) with $x$ and $y$ interchanged: $x = 2^2 \cdot 3 \cdot 5 = 60$ and $y = 2 \cdot 3 \cdot 5 \cdot 7 = 210$.
(iv) $a_2 = 2, b_2 = 1, a_7 = 1, b_7 = 0$, i.e., same as (i) with $x$ and $y$ interchanged: $x = 2^2 \cdot 3 \cdot 5 \cdot 7 = 420$ and $y = 2 \cdot 3 \cdot 5 = 30$.

The assumption that $x > y$, rules out (i) and (iii), so either $x = 210$ and $y = 60$ or $x = 420$ and $y = 30$.

**Problem 4:** Find all integers $x$ such that $0 \leq x \leq 1000$ and

\[
\begin{align*}
x &\equiv 2 \pmod{3}, \\
x &\equiv 6 \pmod{7}, \\
x &\equiv 9 \pmod{10}.
\end{align*}
\]

Explain your answer.

**Solution:** By the Chinese remainder Theorem, the system of congruences

\[
\begin{align*}
x &\equiv 2 \pmod{3}, \\
x &\equiv 6 \pmod{7}, \\
x &\equiv 9 \pmod{10}.
\end{align*}
\]

has a unique solution, modulo $3 \cdot 7 \cdot 10 = 210$. Clearly $x = -1$ is a solution of this system. Thus it is the only solution, modulo 210. In other words, $x = -1 + 210 \cdot n$, where $n$ is an integer. The condition that $0 \leq x \leq 1000$ is satisfied if and only if $n = 1, 2, 3$ or 4. Thus $x = 209, 419$ and $629$ and $839$ are the only solutions in the range $0 \leq x \leq 1000$. 