Extra credit question

The eminent mathematician, Professor Lurve, has a favourite musical instrument: a cymbal with a curious shape. In view of its geometry, the vibrations of this cymbal are conveniently described by the wave equation expressed in terms of the two coordinates \((r, \theta)\):

\[
\frac{\partial^2 u}{\partial t^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right), \quad 0 \leq r \leq 1 \quad 0 \leq \theta \leq \pi
\]

with boundary condition, \(u_r(1, \theta, t) = 0\). Use separation of variables to demonstrate that the solution of the initial-value problem, \(u(r, \theta, 0) = 0\) and \(u_t(r, \theta, 0) = g(r, \theta)\), can be written as the superposition of normal modes,

\[
u = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} A_{nj} \sin(\omega_{nj} t) F_j(r) G_n(\cos \theta),
\]

where you should determine the functions \(F_j(r)\) and \(G_n(\cos \theta)\), along with the normal-mode frequency \(\omega_{nj}\). Provide an expression for the normal-mode amplitudes \(A_{nj}\) in terms of integrals over the initial condition \(g(r, \theta)\). Make this expression as explicit as you can by using the helpful information given below and proving that

\[
\frac{1}{2} [J_1^2(k) + J_0^2(k)] = \int_0^1 r J_0^2(kr) dr.
\]

Using Legendre’s equation, show that

\[
\int_{-1}^{1} P_0(x) dx = 2 \quad \text{and} \quad \int_{-1}^{1} P_n(x) dx = 0 \quad \text{if } n > 0.
\]

Hence, if

\[
\int_0^{\pi} g(r, \theta) \sin \theta d\theta = 0,
\]

argue that the sum over \(n\) starts with \(n = 1\).

Given that \(n\) is so restricted, which of the normal modes has the lowest frequency, and what is that frequency? The pleasantness of sounds from a musical instrument is sometimes attributed to these sounds being composed of a sum of frequencies, \(m\Omega\), with \(m = 1, 2, \ldots\) for some \(\Omega\) (i.e. a harmonic sequence); how pleasant would you imagine Lurve’s cymbal to sound?

Helpful information:

Bessel’s equation is

\[
r^2 y'' + ry' + \left(k^2 r^2 - m^2\right)y = 0,
\]

and has the solution, \(y(r) = J_m(kr)\), which is regular at \(r = 0\). \(J_0(z)\) and \(J_1(z)\) satisfy the relations

\[
\frac{d}{dz} J_0(z) = -J_1(z), \quad \frac{d}{dz} [zJ_1(z)] = zJ_0(z).
\]

Legendre’s equation is

\[
\frac{d}{dx} \left[ (1 - x^2) \frac{dy}{dx} \right] + \lambda y = 0;
\]

the solutions that are regular at \(x = \pm 1\) are \(\lambda = n(n + 1)\) and \(y = P_n(x)\) (the Legendre polynomial of degree \(n\)), with \(n = 0, 1, 2, \ldots\) Also, \(P_n(1) = 1\) and

\[
\int_{-1}^{1} P_n^2(x) dx = \frac{2}{1 + 2n}.
\]
Solution details

Let \( u = T(t)R(r)Y(x) \) with \( x = \cos \theta \). Then

\[
T'' + \omega^2 T = 0, \quad r^2 R'' + r R' + k^2 r^2 R = 0, \quad \frac{d}{dx} \left[ (1 - x^2) \frac{dY}{dx} \right] + \lambda Y = 0,
\]

with \( \omega^2 = k^2 + \lambda \). Thus, given \( u_t(r, \theta, 0) = 0 \) and demanding regularity at \( r = 0 \) and \( x = \pm 1 \), we find that \( T(t), R(r) \) and \( Y(x) \) are given by \( \sin \omega t, J_0(kr) \) and \( P_n(x) \), respectively, along with \( \lambda = n(n + 1) \) and \( n = 0, 1, \ldots \). The boundary condition at \( r = 1 \) implies \( J_0'(k) = -J_1(k) = 0 \), and so \( k = z_j \), the \( j \)th zero of \( J_1(z) \). We now write a general solution,

\[
u = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} c_{nj} J_0(z_j r) P_n(x) \sin \omega_{nj} t, \quad \omega_{nj} = \sqrt{n(n + 1) + z_j^2}.
\]

The remaining initial condition demands

\[
c_{nj} = \frac{2n + 1}{\omega_{nj} [J_0(z_j)]^2} \int_0^1 \int_0^\pi g(r, \theta) J_0(z_j r) P_n(\cos \theta) r \sin \theta \, d\theta \, dr,
\]

in view of the results derived or provided for \( \int_0^1 r [J_0(z_j r)]^2 \, dr \) and \( \int_{-1}^1 [P_n(x)]^2 \, dx \). The \( n = 0 \) term of the sum for \( u \) drops out when \( g \) satisfies the additional integral because \( c_{0j} \) then vanishes (\( \int_{-1}^1 P_0 dx = 2 \) since \( P_0 = 1 \); integrating Legendre’s equation in \( x \) immediately implies \( \int_{-1}^1 P_n dx = 0 \) if \( n \neq 0 \)). Finally, for \( z \gg 1 \), we know that \( J_1(z) \approx \sqrt{2/\pi z} \cos (z - \frac{3\pi}{4}) \), and so \( k_j \approx 2 \pi j + \frac{\pi}{4} \) for \( j \gg 1 \). Thus, although the notes of the cymbal do not form a harmonic sequence in \( j \) for each \( n \), they are close at high frequency, suggesting that the instrument may sound pleasant.