Sample Midterm exam

Closed book exam; no calculators. Adequately explain the steps you take.

1. Use separation of variables to solve

\[ u_{xx} + u_{yy} = 0, \quad \text{in } 0 \leq x \leq \pi, \ 0 \leq y < \infty, \]

subject to

\[ u(0, y) = u(\pi, y) = 0, \quad u(x, 0) = 1, \quad u \to 0 \text{ as } y \to \infty. \]

Show that

\[ u(x, y) = \frac{4}{\pi} \sum_{n > 0, n \text{ odd}} \frac{1}{n} e^{-ny} \sin nx. \]

Using

\[ \frac{1}{2} \ln \left( \frac{1 + \psi}{1 - \psi} \right) = \sum_{n > 0, n \text{ odd}} \frac{\psi^n}{n}, \]

sum the series for \( u(x, y) \), and hence show that

\[ u(x, y) = \frac{1}{i\pi} \ln \left( \frac{\sinh y + i \sin x}{\sinh y - i \sin x} \right). \]

2. Starting with Bessel’s equation with \( m = 0 \), multiply by \( y'(r) \), to show that

\[ \frac{d}{dr} \left[ r^2 (y')^2 \right] + k^2 r^2 \frac{d}{dr} (y^2) = 0 \]

has solution \( y = J_0(kr) \). If \( k_n \) is the \( n^{th} \) zero of \( J_0(z) \), show that

\[ \int_0^1 r |J_0(k_n r)|^2 dr = \frac{1}{2} |J_1(k_n)|^2. \]

Use separation of variables to solve

\[ \frac{1}{r^3} (ru_r)_r + u_{zz} = 0, \quad \text{in } r \leq 1, \ 0 \leq z \leq 1, \]

subject to

\[ u(1, z) = 0, \quad u_z(r, 0) = 0, \quad u(r, 1) = 1, \]

Show that

\[ u(r, z) = \sum_{n=1}^{\infty} \frac{2 J_0(k_n r) \cosh(k_n z)}{k_n J_1(k_n) \cosh k_n}. \]

Helpful information:

Bessel’s equation is

\[ r^2 y'' + r y' + (k^2 r^2 - m^2) y = 0, \]

and has the solution, \( y(r) = J_m(kr) \), which is regular at \( r = 0 \). \( J_0(z) \) and \( J_1(z) \) satisfy the relations

\[ \frac{d}{dz} J_0(z) = -J_1(z), \quad \frac{d}{dz} [z J_1(z)] = z J_0(z). \]